Common Fixed Point Theorems for Multivalued Compatible Maps in Ifms
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ABSTRACT
The aim of this paper is to obtain the notion of multivalued weakly compatible (owc) maps and prove common fixed point theorems for single and multi valued maps by using a contractive condition of integral type in intuitionistic fuzzy metric spaces.
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KeyWords
Weak Compatible maps, weakly compatible maps, Fixed Points and intuitionistic fuzzy metric space.
1. Introduction

The notion of fuzzy sets was introduced by L. A. Zadeh ([8], 1965). In J. H. Park ([6], 2004) introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to A. George and P. Veeramani ([5], 1994). Various authors have worked on this topic and proved some fixed point theorems for various generalizations of contraction mappings in fuzzy metric space. Sedghi. et al. ([7], 2008) established a common fixed point theorem for weakly compatible mappings in intuitionistic fuzzy metric space satisfying a contractive condition of integral type. Note that common fixed point theorems for single and multi-valued maps are interesting and play a major role in many areas.

More recently, Al-Thagafi and N. Shahzad [3] weakened the concept of compatibility by giving a new notion of occasionally weakly compatible (owc) maps which is most general among all the commutativity concepts.

We first give some preliminaries and definitions.

2. Preliminaries

2.1 Definition. A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $*$ is satisfying the following conditions.

(a) $*$ is commutative and associative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0, 1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

2.2 Definition: A binary operation $\circ : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm if $\circ$ it satisfies the following conditions:

(1) $\circ$ is associative and commutative,
(2) $\circ$ is continuous,
(3) $a \circ 0 = a$ for all $a \in [0, 1],$
(4) $a \circ b \leq c \circ d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

2.3 Definition: A 5-tuple $(X, M, N, *, \circ)$ is called an intuitionistic fuzzy metric space if $X$ is an arbitrary (non-empty) set, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm and $M, N$ are fuzzy sets on $X^2 \times (0, \infty)$, satisfying the following conditions: for each $x, y, z \in X$ and $t, s > 0$,

(a) $M(x, y, t) + N(x, y, t) \leq 1$,
(b) $M(x, y, t) > 0$,
Then $(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non nearness between $x$ and $y$ with respect to $t$, respectively.

Example 1. (Induced intuitionistic fuzzy metric space) Let $(X, d)$ be a metric space. Denote $a \ast b = ab$ and $a \diamond b = \min \{1, a+b\}$ for all $a, b \in [0, 1]$ and let $M_d$ and $N_d$ be fuzzy set on $X^2 \times (0, \infty)$, defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then $(M_d, N_d)$ is an intuitionistic fuzzy metric space on $X$. We call this intuitionistic fuzzy metric induced by a metric $d$.

Through out the paper $X$ will represent the intuitionistic fuzzy metric space $(X, M, N, \ast, \diamond)$ and $CB(X)$, the set of all non-empty closed and bounded sub-sets of $X$. For $A, B \in CB(X)$ and for every $t > 0$, denote

$$H(A, B, t) = \sup \{M(a, b, t); a \in A, b \in B\} \quad \text{and} \quad H(A, B, t) = \inf \{N(a, b, t); a \in A, b \in B\}.$$

If $A$ consists of a single point $a$, we write $\delta_M(A, B, t) = M(A, B, t)$ and $\delta_N(A, B, t) = N(A, B, t)$. If $B$ also consists of a single point $b$, we write $\delta_M(A, B, t) = M(A, B, t)$ and $\delta_N(A, B, t) = N(A, B, t)$.

It follows immediately from definition that

$$\delta_M(A, B, t) = \delta_M(B, A, t) \geq 0 \quad \text{and} \quad \delta_N(A, B, t) = \delta_N(B, A, t) \geq 0.$$
\[ \delta_N(A, B, t)=0 \iff A=B=\{a\} \text{ for all } A,B \in \text{CB}(X). \]

**2.4 Definition:** A point \( x \in X \) is called a coincidence point (resp. fixed point) of \( A:X \to X \), \( B:X \to \text{CB}(X) \) if \( Ax \in Bx \) (resp. \( x=Ax=Bx \)).

**2.5 Definition:** Maps \( A:X \to X \) and \( B:X \to \text{CB}(X) \) are said to be compatible if \( ABx \in \text{CB}(X) \) for all \( x \in X \) and \( \lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1 \) and \( \lim_{n \to \infty} N(ABx_n, BAx_n, t) = 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Bx_n \to M \in \text{CB}(x) \) and \( Ax_n \to x \in M \).

**2.6 Definition:** Maps \( A:X \to X \) and \( B:X \to \text{CB}(X) \) are said to be weakly compatible if they commute at coincidence points. i.e., if \( ABx=BAx \) whenever \( Ax=Bx \).

**2.7 Definition:** Maps \( A:X \to X \) and \( B:X \to \text{CB}(X) \) are said to be multivalued weakly compatible (owc) if there exists some point \( x \in X \) such that \( Ax \in Bx \) and \( ABx \subseteq BAx \).

Clearly weakly compatible maps are multivalued weakly compatible (owc). However, the converse is not true in general as shown in the following example.

### 3. Main Result

Now, we prove our main result.

**Theorem 1.** Let \((X,M,N,*,\cdot)\) be a complete intuitionistic fuzzy metric space with continuous t-norm * and continuous t-corm \(\cdot\) defined by \( t^*t=t \) and \((1-t)\cdot(1-t) \leq (1-t) \) for all \( t \in [0,1] \) such that, \( A:X \to X \) and \( B:X \to \text{CB}(X) \) be single and multi valued mappings respectively such that the maps \((A,S)\) and \((B,T)\) are (owc) and satisfy the inequality for all \( x, y \in X \) where \( \phi:[0,1]\to[0,1] \)

\[
(1.1) \quad \int_0^{\delta M(Sx,Ty,kt)} \phi(t)dt > \int_0^{m(x,y,t)} \phi(t)dt
\]

where

\[
m(x,y,t)=\min\{[H(Ax,Sx,t)+H(By,Ty,t)], [M(Ax,By,t)*H(By,Ty,t)*H(By,Sx,(2-\alpha)t)]\},
\]

\[\delta_N(A, B, t)=0 \iff A=B=\{a\} \text{ for all } A,B \in \text{CB}(X), \]
\[ \{[H(Ax, Ty, \alpha t)*H(By, Ty, t)*H(Sx, By, t)]\} \]

\[
\delta N(Sx,Ty,kt) \int_{0}^{\infty} \varphi(t)dt < \int_{0}^{\infty} \phi(t)dt \tag{1.2}
\]

where

\[n(x, y, t)=\max\{[H(Ax, Sx, t)+H(By, Ty,t)], [N(Ax, By, t)\circ H(By, Ty, t)\circ H(By, Sx,(2-\alpha)t)],
[H(Ax, Ty, \alpha t)\circ H(By, Ty, t)\circ H(Sx, By, t)]\}

is a function which is sum able, Lebesque integrable, non-negative and such that \(\int_{0}^{\delta} \phi(t)dt > 0\)

for each \(\epsilon > 0\) for every \(x, y \in X\) and \(t > 0, \alpha \in (0,2)\). Then \(A, B, S\) and \(T\) have unique common fixed point in \(X\).

**Proof.** Since the pairs \((A, S)\) and \((B,T)\) are occasionally weakly compatible (owc) maps, therefore, there exist two elements, \(u, v\) in \(X\) such that \(Au \in Su\), \(ASu \subseteq SAu\)

and \(Bv \in Tv, BTv \subseteq TBv\).

First we prove that \(Au = Bv\).

As \(Au \in Su\) so \(AAu \subseteq SAu, Bv \in Tv\), so \(BBv \subseteq BTv \subseteq TBv\) and hence

\[M(A^2u,B^2v,t) \geq \delta_M(SAu,TBv, t), N(A^2u,B^2v,t) \leq \delta_N(SAu,TBv, t)\]

and if \(Au \neq Bv\) then \(\delta_M(SAu,TBv, t) < 1, \delta_N(SAu,TBv, t) < 1\).

Using (1.2) for \(x = Au, y = Bv\)

\[m(Au, Bv, t)=\min\{[H(AAu, SAu, t)+H(B Bv, TBv,t)],
[M(AAu, B Bv, t)*H(B Bv, TBv, t)*H(SAu, BBv, t)]\}

\[\geq \min\{[M(AAu, SAu, t)+M(B Bv, TBv,t)],
[M(AAu, BBv, t)*M(B Bv, TBv, t)*M(BBv, SAu, (2-\alpha)t)]\}, \tag{1.3}
\]
\[ \begin{align*}
\text{n}(A, B, t) &= \max \{[H(AA, SA, t) + H(BB, TB, t)] ,
\min \{N(AA, BB, t) \land H(BB, TB, t) \land H(BB, SA, (2-\alpha)t)\},
\min \{H(AA, TB, (2-\alpha)t) \land H(BB, TB, t) \land H(SA, BB, t)\}\}
\end{align*} \]

Since, * and ◊ is continuous, letting \( \alpha \to 1 \) in (1.3) and (1.4), we get
\[ \begin{align*}
m(A, B, t) &\geq \min \{[M(A^2, SA, t) + M(B^2, TB, t)],
\min \{[M(A^2, B^2, t) \land M(B^2, TB, t) \land M(B^2, SA, t)]\},
\min \{[\delta_M(SA, TB, t) \land 1 \land \delta_M(SA, TB, t)]\}\}
\end{align*} \]

\[ \begin{align*}
\text{n}(A, B, t) &\leq \max \{[N(A^2, SA, t) + N(B^2, TB, t)],
\max \{N(A^2, BB, t) \lor N(B^2, TB, t) \lor N(B^2, SA, t)\},
\max \{N(A^2, TB, t) \lor N(B^2, TB, t) \lor N(SA, B^2, t)\}\}
\end{align*} \]

From (1.1) and (1.5), (1.2) and (1.6), we have
\[ \begin{align*}
\delta M(SA, TB, kt) &= \int_0^\infty \phi(t)dt > \int_0^\infty \phi(t)dt \geq \int_0^\infty \phi(t)dt
\end{align*} \]

\[ \begin{align*}
\delta N(SA, TB, kt) &= \int_0^\infty \phi(t)dt < \int_0^\infty \phi(t)dt \leq \int_0^\infty \phi(t)dt , \text{ a contradiction.}
\end{align*} \]

Hence \( A = B \).

Also \( M(A^2, Bu, t) \geq \delta_M(SA, Tu, t), \ N(A^2, Bu, t) \leq \delta_N(SA, Tu, t), \)
\( M(A^2, Tu, t) \geq \delta_M(SA, Tu, t), \ N(A^2, Tu, t) \leq \delta_N(SA, Tu, t), \)
Now, we claim that $Au = u$. It not, then

$$\delta_M(SAu, Tu, t) < 1, \delta_N(SAu, Tu, t) < 1$$

Considering (1.1) and (1.2) for $Au = x, u = y, \alpha = 1$

$$m(Au, u, t) = \min\{[H(AAu, SAu, t) + H(Bu, Tu, t)],\]

$$[M(AAu, Bu, t) * H(Bu, Tu, t) * H(Bu, SAu, t)],$$

$$[H(AAu, Tu, t) * H(Bu, Tu, t) * H(SAu, Bu, t)]\}$$

$$= \min\{[H(A^2u, SAu, t) + H(Bu, Tu, t)],\]

$$[M(A^2u, Bu, t) * H(Bu, Tu, t) * H(Bu, SAu, t)],$$

$$[H(A^2u, Tu, t) * H(Bu, Tu, t) * H(SAu, Bu, t)]\}$$

$$\geq \min\{[M(A^2u, SAu, t) + M(Bu, Tu, t)],\]

$$[M(A^2u, Bu, t) * M(Bu, Tu, t) * M(Bu, SAu, t)],$$

$$[M(A^2u, Tu, t) * M(Bu, Tu, t) * M(SAu, Bu, t)]\}$$

$$\geq \min\{[1 + 1], [\delta_M(SAu, Tu, t) * 1 * \delta_M(Tu, SAu, t)],\]

$$[\delta_M(SAu, Tu, t) * 1 * \delta_M(SAu, Tu, t)]\}

$$m(Au, u, t) \geq \delta_M(SAu, Tu, t) \quad (1.7)$$

$$n(Au, u, t) = \max\{[H(AAu, SAu, t) + H(Bu, Tu, t)],\}

$$[N(AAu, Bu, t) \diamond H(Bu, Tu, t) \diamond H(Bu, SAu, t)],$$

$$[H(AAu, Tu, t) \diamond H(Bu, Tu, t) \diamond H(SAu, Bu, t)]\}$$

$$= \max\{[H(A^2u, SAu, t) + H(Bu, Tu, t)],\]

$$[N(A^2u, Bu, t) \diamond H(Bu, Tu, t) \diamond H(Bu, SAu, t)],$$

$$[H(A^2u, Tu, t) \diamond H(Bu, Tu, t) \diamond H(SAu, Bu, t)]\}$$

$$\leq \max\{[N(A^2u, SAu, t) + N(Bu, Tu, t)],\]

$$[N(A^2u, Bu, t) \diamond N(Bu, Tu, t) \diamond N(Bu, SAu, t)],$$

$$[N(A^2u, Tu, t) \diamond N(Bu, Tu, t) \diamond N(SAu, Bu, t)]\}$$

$$\leq \max\{[0 + 0], [\delta_N(SAu, Tu, t) \diamond 0 \diamond \delta_N(Tu, SAu, t)],\]

$$[\delta_N(SAu, Tu, t) \diamond 0 \diamond \delta_N(SAu, Tu, t)]\}

$$m(Au, u, t) \geq \delta_N(SAu, Tu, t) \quad (1.8)$$

From (1.1) and (1.7), (1.2) and (1.8) we have

$$\delta_M(SAu, Tu, kt) \quad m(Au, u, t) \quad \delta_M(SAu, Tu, t)$$

$$\int_0^t \phi(t)dt > \int_0^t \phi(t)dt \geq \int_0^t \phi(t)dt$$
\[ \delta N(SA_u, Tu, kt) \quad n(A_u, t) \quad \delta N(SA_u, Tu, t) \]
\[ \int_0^\infty \phi(t)dt < \int_0^\infty \phi(t)dt \geq \int_0^\infty \phi(t)dt \]

which is again a contradiction and hence \( A = u \).

Similarly, we can get \( B_v = v \).

Thus A, B, S and T have a common fixed point in X.

For uniqueness let \( u \neq u' \) be another fixed point of A, B, S and T, then (1.1) and (1.2) gives

\[ m(u, u', t) = \min \{ [H(Au, Su, t) + H(Bu', Tu', t)], \]
\[ [M(Au, Bu', t) * H(Bu', Tu', t) * H(Bu', Su, (2-\alpha)t)], \]
\[ [H(Au, Tu', \alpha t) * H(Bu', Tu', t) * H(Su, Bu', t)] \}

\[ n(u, u', t) = \max \{ [H(Au, Su, t) + H(Bu', Tu', t)], \]
\[ [M(Au, Bu', t) * H(Bu', Tu', t) * H(Bu', Su, (2-\alpha)t)], \]
\[ [H(Au, Tu', \alpha t) * H(Bu', Tu', t) * H(Su, Bu', t)] \}

Letting \( \alpha \rightarrow 1 \)

\[ m(u, u', t) = \min \{ [\delta M(Au, Su, t) + \delta M(Bu', Tu', t)], \]
\[ [M(Au, Bu', t) * \delta M(Bu', Tu', t) * \delta M(Bu', Su, t)], \]
\[ [\delta M(Au, Tu', \alpha t) * \delta M(Bu', Tu', t) * \delta M(Su, Bu', t)] \}

\[ n(u, u', t) = \max \{ [\delta N(Au, Su, t) + \delta N(Bu', Tu', t)], \]
\[ [N(Au, Bu', t) * \delta N(Bu', Tu', t) * \delta N(Bu', Su, t)], \]
\[ [\delta N(Au, Tu', \alpha t) * \delta N(Bu', Tu', t) * \delta N(Su, Bu', t)] \}

Again from (1.1) and (1.9), (1.2) and (1.10) we obtain

\[ \delta M(Su, Tu', kt) \quad m(u, u', t) \quad \delta M(Su, Tu', t) \]
\[ \int_0^\infty \phi(t)dt > \int_0^\infty \phi(t)dt \geq \int_0^\infty \phi(t)dt \]
\[ \delta N(Su, Tu', kt) = \int_0^{n(u, u', t)} \varphi(t)dt < \int_0^{n(u, u', t)} \delta N(Su, Tu', t) \]

Which yields \( Su = Tu \), i.e., \( u = u' \).

Thus, A, B, S and T have unique common fixed point.

Now, we furnish our theorem with an example.

**Example 3.** Let \((X, M, N, *, \phi)\) be a complete intuitionistic fuzzy metric space \( X = \mathbb{R}^+ \) with \( a*b = \min \{ a, b \} \) and \( a\oplus b = \min\{1, a+b\} \) for all \( a, b \in [0,1] \), and

\[
M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)} \quad \text{for all } t > 0
\]

Define the maps \( A : X \rightarrow X \) and \( B : X \rightarrow CB(X) \) by setting

\[
A(x) = \begin{cases} 
2x - 1, & x \leq 5 \\
2x, & x > 5
\end{cases}, \quad B(x) = \begin{cases} 
3 - 2x, & x \leq 1 \\
[x + 1, x > 1]
\end{cases}
\]

\[
S(x) = \begin{cases} 
\{1\}, & x < 2 \\
[2x, 2x + 5], & x \geq 2
\end{cases}, \quad T(x) = \begin{cases} 
\{1\}, & x = 1 \\
[x, x + 2], & \text{otherwise}
\end{cases}
\]

Here the pairs \((A, S)\) and \((B, T)\) are owc.

Define \( \varphi : [0,1] \rightarrow [0,1] \) as \( \varphi(0) = 0 \), \( \varphi(1) = 1 \) and \( \varphi(s) = \sqrt{s} \) for \( 0 < s < 1 \), then the contractive condition (1.1) is satisfied for all \( t > 0 \). Thus all the conditions of our theorem are satisfied and \( '1' \) is the common fixed point of A, B, S and T.

**References**


