THE PRINCIPAL INVARIANTS OF $K_{pq}$ FOR DE-SITTER SPACE-TIME IN SPHERICALLY SYMMETRIC COORDINATE SYSTEM IN A NARROW SENSE OF $V_5$.

Bagde P. O., Thomas K. T.

Author: Mr. P. O. Bagde
Assistant Professor, Mathematics Department, RCOEM, Nagpur, India,
Mob. No. +91 9011000806 E-mail: prafulla.bagde@gmail.com
Co-Author: Dr. K. T. Thomas,
Principal, SFS College, Seminary Hills, Nagpur, India.

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ABSTRACT
The ten principal invariants of $K_{pq}$ for de-sitter space-time in spherically symmetric (s.s.) coordinate system in a Narrow sense of $V_5$

$$ds^2 = -\left(1 - r^2k^2\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) + \left(1 - r^2k^2\right)dt^2 - \left(1 - r^2k^2\right)du^2$$

(1)

are $-k^2$.

Further we reduce line element (1) by taking $k \to 0$ to Minkoski space-time in s.s. coordinate system $V_5$ in a Narrow sense.
1. INTRODUCTION:

According to Takeno, the space time for $V_4$ with metric

$$ds^2 = g_{ij}dx^idx^j$$

spherically symmetric if

$$L_\xi g_{ij} = 0$$  \hspace{1cm} (1.2)

where $L_\xi$ denotes the Lie derivative with respect to the Killing vector $\xi^i$. Takeno [2] has obtain the most general form of the s.s. line in the spherical polar coordinates $(r, \theta, \phi, t)$ as

$$ds^2 = -A dr^2 - B\left(d\theta^2 + \sin^2 \theta d\phi^2\right) + C dt^2 + 2D dr dt$$  \hspace{1cm} (1.3)

where A, B, C, and D are functions of $r$ and $t$.

Takeno [2] has obtained the six principal invariants $\left(-k^2,-k^2,-k^2,-k^2,-k^2,-k^2\right)$ of $K_{pq}$ for de-sitter space time

$$ds^2 = -(1-r^2k^2)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2 \theta d\phi^2\right) + \left(1-r^2k^2\right)dt^2$$  \hspace{1cm} (1.4)

from a symmetric tensor of six dimensional representations of four dimensional quantities by taking

$$K_{ijkl} = K_{\alpha\beta\gamma\delta}$$

where $\alpha \equiv (i,j), \beta \equiv (k,l)$ and the indices $\alpha, \beta \ldots$ run from 1 to 6.

Karade and Thomas [4] have obtained the most general form of the s.s. line element in $V_5$ as

$$ds^2 = -A dr^2 - B\left(d\theta^2 + \sin^2 \theta d\phi^2\right) + C dt^2 - D du^2 + 2E dr dt + 2F dr du + 2G dt du$$  \hspace{1cm} (1.5)

where A, B, C, D, E, F and G are the functions of $r$, $t$ and $u$ and $x^i = (r, \theta, \phi, t, u)$.

Pokely and Thomas [1] have reduced the line element (1.5) in the s.s. coordinate system in a Narrow sense as

$$ds^2 = -A dr^2 - B\left(d\theta^2 + \sin^2 \theta d\phi^2\right) + C dt^2 - D du^2$$  \hspace{1cm} (1.6)

by transformation method and then obtained the Christoffel symbols, Ricci tensors, Curvature tensors etc. for (1.6).

In this paper we shall obtain the ten principal invariants of $K_{pq}$ for de-sitter space time using s.s. line element (1.6). We shall denote that curvature tensor is symmetric tensor $K_{ijkl} = K_{pq}$ \hspace{1cm} [where, $p = (i,j), q = (k,l)$] of ten dimensional representations of five dimensional quantities. The indices $p, q \ldots$ run from 1 to 10 where $1 = (1,2), 2 = (1,3), \ldots, 10 = (4,5)$. Further we shall obtain the ten principal invariants of Minkowski space time in s.s. coordinate system in a Narrow sense of $V_5$.

2. LINE ELEMENT OF DE-SITTER SPACE TIME IN S.S. COORDINATE SYSTEM IN A NARROW SENSE OF $V_5$:

**THEOREM:** The line element of de-sitter space in s.s. coordinate system in a Narrow sense of $V_5$ is

$$ds^2 = -(1-r^2k^2)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2 \theta d\phi^2\right) + \left(1-r^2k^2\right)dt^2$$  \hspace{1cm} (2.1)

**PROOF:** From line element (1.6), we have

$$g_{11} = -A, \quad g_{22} = -B, \quad g_{33} = -B \sin^2 \theta, \quad g_{44} = C, \quad g_{55} = -D$$  \hspace{1cm} (2.2)
\[ g^{11} = -\frac{1}{A}, \quad g^{22} = -\frac{1}{B}, \quad g^{33} = -\frac{1}{B \sin^2 \theta}, \quad g^{44} = \frac{1}{C}, \quad g^{55} = -\frac{1}{D} \] (2.3)

The remaining \( g^{ij} = 0 \)

Pokley and Thomas [1] have obtained the non-vanishing independent components of the curvature tensor \( K_{ijkl} \) of the s.s. space time in a narrow sense of \( V_5 \) with the metric (1.6) given by

\[
\begin{align*}
K_{1212} &= \frac{K_{1313}}{\sin^2 \theta} = f_1, \quad K_{1414} = f_2, \quad K_{1515} = f_3, \quad K_{1224} = \frac{K_{1334}}{\sin^2 \theta} = f_4, \\
K_{1225} &= \frac{K_{1335}}{\sin^2 \theta} = f_5, \quad K_{1415} = f_6, \quad K_{1445} = f_7, \quad K_{1545} = f_8, \quad K_{4545} = f_{13}, \\
K_{2424} &= \frac{K_{3434}}{\sin^2 \theta} = f_9, \quad K_{2525} = \frac{K_{4535}}{\sin^2 \theta} = f_{10}, \quad K_{2425} = \frac{K_{3435}}{\sin^2 \theta} = f_{11}, \quad K_{2323} = f_{12}
\end{align*}
\]

(2.4)

where \( f_1, f_2, \ldots, f_{13} \) are functions of \( r, t \) and \( u \) given by

\[
\begin{align*}
f_1 &= B^* - \frac{B^2 - 2}{4B} - \frac{1}{4} \left( \frac{\dot{A} B'}{A} + \frac{\dot{B} D}{D} - \frac{\dot{A} B}{A} - \frac{\dot{B} D}{D} \right), \\
f_2 &= \frac{\dot{A} - C'}{2} - \frac{1}{4} \left( \frac{\dot{A} - C'}{2} - \frac{\dot{A} - C'}{2} - \frac{\dot{A} - C'}{2} - \frac{\dot{A} - C'}{2} \right).
\end{align*}
\]

(2.5)

Here a prime, a dot and a cap means derivatives with respect to \( r, t \) and \( u \) respectively. Now, the de-sitter space time is of constant curvature and characterized by the equation

\[ K_{ijkl} = k^2 \left( g_{im} g_{jl} - g_{il} g_{jm} \right) \] (2.6)

Therefore, using (2.4) and (2.6), the non-vanishing independent component of the curvature tensor are

\[
\begin{align*}
K_{1212} &= \frac{K_{1313}}{\sin^2 \theta} = -k^2 A B, \quad K_{1414} = k^2 A C, \quad K_{1515} = -k^2 A D, \quad K_{4545} = k^2 C D, \\
K_{2424} &= \frac{K_{3434}}{\sin^2 \theta} = k^2 B C, \quad K_{2525} = \frac{K_{4535}}{\sin^2 \theta} = -k^2 B D, \quad K_{2323} = -k^2 B^2
\end{align*}
\]

(2.7)

Using (2.5) and (2.7) we have
\[-k^2 AB = \frac{B^2}{2} - \frac{B'^2}{4B} - \frac{1}{4} \left( \frac{A'B'}{A} + \frac{\dot{A'B}}{C} - \frac{\ddot{A'B}}{D} \right), \quad k^2 BC = \frac{\dot{B}}{2} - \frac{1}{4} \left( \frac{\dot{B}^2}{B} + \frac{B'C'}{A} + \frac{\dot{B'}C}{C} + \frac{\ddot{B'C}}{D} \right) \]

\[
k^2 AC = \frac{\dot{A}' - C^2}{2} - \frac{1}{4} \left( \frac{\dot{A}'^2}{A} - \frac{C'^2}{C} - \frac{A'C'}{A} + \frac{\dot{A'C}}{C} + \frac{\ddot{A'C}}{D} \right) \]

\[
k^2 BD = \frac{\dot{B}' + D^2}{2} - \frac{1}{4} \left( \frac{\dot{B}'^2}{B} - \frac{B'D'}{A} + \frac{\dot{B'D}}{C} + \frac{\ddot{B'D}}{D} \right) \]

\[
k^2 BC = -B + \frac{1}{4} \left( \frac{B'^2}{A} - \frac{B'^2}{C} + \frac{\dot{B'^2}}{D} \right), \quad k^2 AD = \frac{\dot{A} + D^2}{2} - \frac{1}{4} \left( \frac{\dot{A}^2}{A} + \frac{D'^2}{D} + \frac{A'D'}{A} + \frac{\dot{A'D}}{C} + \frac{\ddot{A'D}}{D} \right) \]

\[
k^2 CD = \frac{\dot{D} - \dot{C}}{2} + \frac{1}{4} \left( \frac{\dot{C}^2}{C} - \frac{C'^2}{D} - \frac{C'D'}{A} - \frac{\dot{C'D}}{C} + \frac{\ddot{C'D}}{D} \right) \]

We obtain values of A, B, C and D satisfying (2.7) as

\[A = \left(1 - r^2 k^2 \right)^{-1}, \quad B = r^2, \quad C = \left(1 - r^2 k^2 \right), \quad D = \left(1 - r^2 k^2 \right) \]

Therefore (1.6) become

\[ds^2 = -(1 - r^2 k^2)^{-1} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + (1 - r^2 k^2) dt^2 - \left(1 - r^2 k^2 \right) du^2 \]

Thus the theorem is proved.

3. RICCI TENSOR \( K_{ij}^{kl} \)

**THEOREM:** The Ricci tensors \( K_{ij}^{kl} \) of de-sitter space time \( V_5 \) in a Narrow sense with the metric (2.1) are

\[
\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_19 = -k^2, \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = \alpha_{16} = \alpha_{17} = \alpha_{18} = 0 \]

where \( \alpha_1, \alpha_2, \ldots, \alpha_{19} \) are the mixed components of \( K_{ij}^{kl} \) of the de-sitter space time \( V_5 \) in a Narrow sense.

**PROOF:** Consider \( K_{ij}^{kl} = g^{ka} g^{lb} K_{ij ab} \)

Using the equations (2.3) and (2.6), the mixed components of \( K_{ij}^{kl} \) are given by
\[\alpha_1 = K_{12}^{12} = K_{13}^{13} = \frac{1}{AB} K_{1212} = -k^2, \quad \alpha_2 = K_{24}^{24} = K_{34}^{34} = -\frac{1}{BC} K_{2424} = -k^2,\]
\[\alpha_3 = K_{14}^{14} = -\frac{1}{AC} K_{1414} = -k^2, \quad \alpha_4 = K_{15}^{15} = \frac{1}{AD} K_{1515} = -k^2\]
\[\alpha_5 = K_{23}^{23} = \frac{1}{B^2} \sin^2 \theta K_{2323} = -k^2, \quad \alpha_6 = K_{25}^{25} = K_{35}^{35} = -\frac{1}{BD} K_{2525} = -k^2\]
\[\alpha_7 = K_{12}^{24} = K_{13}^{34} = -\frac{1}{BC} K_{1224} = 0, \quad \alpha_8 = K_{24}^{12} = K_{34}^{13} = \frac{1}{AB} K_{1224} = 0\]
\[\alpha_9 = K_{12}^{12} = K_{13}^{13} = \frac{1}{BD} K_{1225} = 0, \quad \alpha_{10} = K_{24}^{12} = K_{34}^{13} = \frac{1}{AB} K_{1225} = 0\]
\[\alpha_{11} = K_{24}^{25} = K_{34}^{35} = \frac{1}{BD} K_{2425} = 0, \quad \alpha_{12} = K_{25}^{24} = K_{35}^{34} = -\frac{1}{BC} K_{2425} = 0\]
\[\alpha_{13} = K_{14}^{14} = \frac{1}{AD} K_{1415} = 0, \quad \alpha_{14} = K_{15}^{15} = -\frac{1}{AC} K_{1415} = 0\]
\[\alpha_{15} = K_{14}^{45} = \frac{1}{CD} K_{1445} = 0, \quad \alpha_{16} = K_{45}^{15} = \frac{1}{AD} K_{1445} = 0\]
\[\alpha_{17} = K_{14}^{45} = \frac{1}{CD} K_{1445} = 0, \quad \alpha_{18} = K_{45}^{14} = -\frac{1}{AC} K_{1445} = 0, \quad \alpha_{19} = K_{45}^{15} = -\frac{1}{CD} K_{4545} = -k^2\]  

Thus, above equations prove the theorem.

4. PRINCIPAL INVARIANTS OF $K_{pq}$

**THEOREM:** The ten principal invariants of $K_{pq}$ for de-Sitter space-time with line element (2.1) are $-k^2$.

**PROOF:** We consider curvature tensor is a symmetric tensor by taking

\[K_{ijkl} = K_{pq} \quad [\text{where } p = (i,j), q = (k,l)] \]  

(4.1)

of a ten dimensional representation of five dimensional quantities for the line element (2.1).

The indices $p, q \ldots$ run from 1 to 10, where

\[\{1 = (12), 2 = (13), 3 = (14), 4 = (15), 5 = (23), 6 = (24), 7 = (25), 8 = (34), 9 = (35), 10 = (45)\}\]  

(4.2)

(we will represent 10 by 0, for rest of calculations)

The non-vanishing Ricci tensors $K_{pq} = K_{ijkl} = g_{qm} K_{pm}$ can be expressed in terms of $\alpha_1, \alpha_2, \ldots, \alpha_{19}$ (for a ten dimensional representation of five dimensional quantities) are

\[K_1 = K_{12}^{12} = K_{13}^{13} = \alpha_1, \quad K_2 = K_{14}^{14} = \alpha_2, \quad K_3 = K_{15}^{15} = \alpha_3, \quad K_4 = K_{23}^{23} = \alpha_4, \quad K_5 = K_{24}^{24} = \alpha_5, \quad K_6 = K_{34}^{34} = \alpha_6, \quad K_7 = K_{35}^{35} = \alpha_7, \quad K_8 = K_{45}^{45} = \alpha_8, \quad K_9 = K_{12}^{12} = \alpha_9, \quad K_{10} = K_{13}^{13} = \alpha_{10}, \quad K_{11} = K_{14}^{14} = \alpha_{11}, \quad K_{12} = K_{15}^{15} = \alpha_{12}, \quad K_{13} = K_{23}^{23} = \alpha_{13}, \quad K_{14} = K_{24}^{24} = \alpha_{14}, \quad K_{15} = K_{34}^{34} = \alpha_{15}, \quad K_{16} = K_{35}^{35} = \alpha_{16}, \quad K_{17} = K_{45}^{45} = \alpha_{17}, \quad K_{18} = K_{25}^{25} = \alpha_{18}, \quad K_{19} = K_{36}^{36} = \alpha_{19}\]  

(4.3)

Using equations (4.3), the ten principal invariants $\lambda_{pq}$ of $K_{pq}$ are given as solution from the equation

\[\det \left( K_{pq} - \lambda \delta_{pq} \right) = 0 \quad \text{where} \quad \delta_{pq} = 1, \quad p = q; \quad \delta_{pq} = 0, \quad p \neq q\]  

(4.4)

i.e.
\[
\begin{vmatrix}
\alpha_1 - \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 - \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_2 - \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_3 - \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_4 - \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_5 - \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_6 - \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 - \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_8 - \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix}
= 0
\]

\[\Rightarrow (\alpha_1 - \lambda)^2 (\alpha_2 - \lambda) (\alpha_3 - \lambda) (\alpha_4 - \lambda) (\alpha_5 - \lambda)^2 (\alpha_6 - \lambda)^2 (\alpha_7 - \lambda) (\alpha_8 - \lambda) = 0\]

Using equation (3.1), we have

\[\left( -k^2 - \lambda \right)^0 = 0\]

\[\Rightarrow \{\lambda\} \equiv \lambda^{\text{s}} = \left( -k^2, -k^2, -k^2, -k^2, -k^2, -k^2, -k^2, -k^2 \right) \]  \hspace{1cm} (4.5)

Thus the theorem is proved.

Further if we take \( k \rightarrow 0 \) then (2.1) becomes the line element of the Minkowski space-time in s.s. coordinate system in a Narrow sense of \( V^5 \), i.e.

\[ds^2 = -dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + dt^2 - du^2\] \hspace{1cm} (4.6)

All of \( \lambda^{\text{s}} \) (i.e. \( \alpha_1, \alpha_2, \ldots, \alpha_{19} \)) become 0 and ten principal invariants \( \lambda^{\text{s}} \) become 0.

5. CONCLUSION:

Using [3] we can state that, if all ten principal invariants \( \lambda^{\text{s}} \) are equal to constant \(-k^2 (\neq 0)\) or 0, then the spherically symmetric space-time \( V^5 \) in Narrow sense is de-sitter space-time \( V^5 \) or Minkowski space-time \( V^5 \) respectively in Narrow sense. The spherically symmetric line element in a Canonical coordinate system is given respectively by (2.1) or (4.6) to within a transformation of t.

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References


