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THE PRINCIPAL INVARIANTS OF K_{pq} FOR DE-SITTER SPACE-TIME IN SPHERICALLY SYMMETRIC COORDINATE SYSTEM IN A NARROW SENSE OF V_5.

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ABSTRACT

The ten principal invariants of K_{pq} for de-sitter space-time in spherically symmetric (s.s.) coordinate system in a Narrow sense of V_5

$$ds^2 = -\left(1-r^2k^2\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \left(1-r^2k^2\right)dt^2 - \left(1-r^2k^2\right)du^2 \qquad (1)$$

are $-k^2$.

Further we reduce line element (1) by taking $k \to 0$ to Minkoski space-time in s.s. coordinate system V_5 in a Narrow sense.

1. INTRODUCTION:

According to Takeno, the space time for V_4 with metric

$$ds^2 = g_{ij} dx^i dx^j \tag{1.1}$$

spherically symmetric if

$$L_\xi \, g_{ij} = 0 \tag{1.2}$$

where L_ξ denotes the Lie derivative with respect to the Killing vector ξ^i.

Takeno [2] has obtain the most general form of the s.s. line in the spherical polar coordinates (r, θ, ϕ, t) as

$$ds^2 = -A dr^2 - B\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) + C\, dt^2 + 2D\, dr\, dt \tag{1.3}$$

where A, B, C, and D are functions of r and t.

Takeno [2] has obtained the six principal invariants $\left(-k^2, -k^2, -k^2, -k^2, -k^2, -k^2\right)$ of K_{pq} for de-sitter space time

$$ds^2 = -\left(1 - r^2k^2\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) + \left(1 - r^2k^2\right) dt^2 \tag{1.4}$$

from a symmetric tensor of six dimensional representations of four dimensional quantities by taking $K_{ijkl} \equiv K_{\alpha\beta}$ where $\alpha \equiv (i\, j)$, $\beta \equiv (k\, l)$ and the indices α, β . . . run from 1 to 6.

Karade and Thomas [4] have obtained the most general form of the s.s. line element in V_5 as

$$\begin{aligned} ds^2 = {} & -A\, dr^2 - B\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) + C\, dt^2 - D\, du^2 + 2E\, dr\, dt \\ & + 2F\, dr\, du + 2G\, dt\, du \end{aligned} \tag{1.5}$$

where A, B, C, D, E, F and G are the functions of r, t and u and $x^i = (r, \theta, \phi, t, u)$.

Pokely and Thomas [1] have reduced the line element (1.5) in the s.s. coordinate system in a Narrow sense as

$$ds^2 = -A\, dr^2 - B\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) + C\, dt^2 - D\, du^2 \tag{1.6}$$

by transformation method and then obtained the Christoffel symbols, Ricci tensors, Curvature tensors etc. for (1.6).

In this paper we shall obtain the ten principal invariants of K_{pq} for de-sitter space time using s.s. line element (1.6). We shall denote that curvature tensor is symmetric tensor $K_{ijkl} \equiv K_{pq}$ [where, $p \equiv (i\, j)$, $q \equiv (k\, l)$] of ten dimensional representations of five dimensional quantities. The indices p, q . . . run from 1 to 10 where $1 \equiv (12)$, $2 \equiv (13), \ldots, 10 \equiv (45)$. Further we shall obtain the ten principal invariants of Minkoski space time in s.s. coordinate system in a Narrow sense of V_5

2. LINE ELEMENT OF DE-SITTER SPACE TIME IN S.S. COORDINATE SYSTEM IN A NARROW SENSE OF V_5:

THEOREM: The line element of de-sitter space in s.s. coordinate system in a Narrow sense of V_5 is

$$ds^2 = -\left(1 - r^2k^2\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) + \left(1 - r^2k^2\right) dt^2 - \left(1 - r^2k^2\right) du^2 \tag{2.1}$$

PROOF: From line element (1.6), we have

$$g_{11} = -A, \; g_{22} = -B, \; g_{33} = -B\sin^2\theta, \; g_{44} = C, \; g_{55} = -D \tag{2.2}$$

$$\left.\begin{aligned} &\text{and } g^{11} = -\frac{1}{A},\ g^{22} = -\frac{1}{B},\ g^{33} = -\frac{1}{B\sin^2\theta},\ g^{44} = \frac{1}{C},\ g^{55} = -\frac{1}{D} \\ &\qquad\qquad \text{the remaining } g^{ij} = 0 \end{aligned}\right\} \tag{2.3}$$

Pokley and Thomas [1] has obtain the non-vanishing independent components of the curvature tensor K_{ijkl} of the s.s. space time in a Narrow sense of V_5 with the metric (1.6) given by

$$\left.\begin{aligned} &K_{1212} = \frac{K_{1313}}{\sin^2\theta} = f_1\ , K_{1414} = f_2\ ,\ K_{1515} = f_3\ , K_{1224} = \frac{K_{1334}}{\sin^2\theta} = f_4\ , \\ &K_{1225} = \frac{K_{1335}}{\sin^2\theta} = f_5\ , K_{1415} = f_6\ , K_{1445} = f_7\ , K_{1545} = f_8\ , K_{4545} = f_{13}\ , \\ &K_{2424} = \frac{K_{3434}}{\sin^2\theta} = f_9\ , K_{2525} = \frac{K_{3535}}{\sin^2\theta} = f_{10}\ , K_{2425} = \frac{K_{3435}}{\sin^2\theta} = f_{11}\ , \frac{K_{2323}}{\sin^2\theta} = f_{12} \end{aligned}\right\} \tag{2.4}$$

where $f_1,\ f_2\ , \ldots , f_{13}$ are functions of r, t and u given by

$$\left.\begin{aligned} &f_1 = \frac{B''}{2} - \frac{B'^2}{4B} - \frac{1}{4}\left(\frac{A'B'}{A} + \frac{\dot{A}\dot{B}}{C} - \frac{\hat{A}\hat{B}}{D}\right),\ f_2 = \frac{\ddot{A} - C''}{2} - \frac{1}{4}\left(\frac{\dot{A}^2}{A} - \frac{C'^2}{C} - \frac{A'C'}{A} + \frac{\dot{A}\dot{C}}{C} + \frac{\hat{A}\hat{C}}{D}\right) \\ &f_3 = \frac{\hat{\hat{A}} + D''}{2} - \frac{1}{4}\left(\frac{\hat{A}^2}{A} + \frac{D'^2}{D} + \frac{A'D'}{A} + \frac{\dot{A}\dot{D}}{C} + \frac{\hat{A}\hat{D}}{D}\right),\quad f_4 = -\frac{\dot{B}'}{2} + \frac{1}{4}\left(\frac{\dot{A}B'}{A} + \frac{\dot{B}C'}{C} + \frac{B'\dot{B}}{B}\right) \\ &f_5 = -\frac{\hat{B}'}{2} + \frac{1}{4}\left(\frac{\hat{A}B'}{A} + \frac{\hat{B}D'}{D} + \frac{B'\hat{B}}{B}\right),\quad f_6 = \frac{\hat{\dot{A}}}{2} - \frac{1}{4}\left(\frac{\dot{A}\hat{A}}{A} + \frac{\dot{A}\hat{C}}{C} - \frac{\hat{A}\dot{D}}{D}\right) \\ &f_7 = \frac{\hat{C}'}{2} - \frac{1}{4}\left(\frac{C'\hat{A}}{A} + \frac{D'\hat{C}}{D} + \frac{C'\hat{C}}{C}\right),\quad f_8 = \frac{\dot{D}'}{2} - \frac{1}{4}\left(\frac{D'\dot{D}}{D} + \frac{\dot{A}D'}{A} + \frac{C'\dot{D}}{C}\right) \\ &f_9 = \frac{\ddot{B}}{2} - \frac{1}{4}\left(\frac{\dot{B}^2}{B} + \frac{B'C'}{A} + \frac{\dot{B}\dot{C}}{C} + \frac{\hat{B}\hat{C}}{D}\right),\quad f_{10} = \frac{\hat{\hat{B}}}{2} - \frac{1}{4}\left(\frac{\hat{B}^2}{B} - \frac{B'D'}{A} + \frac{\dot{B}\dot{D}}{C} + \frac{\hat{B}\hat{D}}{D}\right) \\ &f_{11} = \frac{\hat{\dot{B}}}{2} - \frac{1}{4}\left(\frac{\dot{B}\hat{B}}{B} + \frac{\dot{B}\hat{C}}{C} + \frac{\hat{B}\dot{D}}{D}\right),\quad f_{12} = -B + \frac{1}{4}\left(\frac{B'^2}{A} - \frac{\dot{B}^2}{C} + \frac{\hat{B}^2}{D}\right) \\ &f_{13} = \frac{\ddot{D} - \hat{\hat{C}}}{2} + \frac{1}{4}\left(\frac{\hat{C}^2}{C} - \frac{\dot{D}^2}{D} - \frac{C'D'}{A} - \frac{\dot{C}\dot{D}}{C} + \frac{\hat{C}\hat{D}}{D}\right) \end{aligned}\right\} \tag{2.5}$$

Here a prime, a dot and a cap means derivatives with respect to r, t and u respectively.
Now, the de-sitter space time is of constant curvature and characterized by the equation

$$K_{ijkl} = k^2\left(g_{im}g_{jl} - g_{il}g_{jm}\right) \tag{2.6}$$

Therefore, using (2.4) and (2.6), the non-vanishing independent component of the curvature tensor are

$$\left.\begin{aligned} &K_{1212} = \frac{K_{1313}}{\sin^2\theta} - k^2AB\ ,\ K_{1414} = k^2AC\ ,\ K_{1515} = -k^2AD\ , K_{4545} = k^2CD, \\ &K_{2424} = \frac{K_{3434}}{\sin^2\theta} = k^2BC, K_{2525} = \frac{K_{3535}}{\sin^2\theta} = -k^2BD, \frac{K_{2323}}{\sin^2\theta} = -k^2B^2 \end{aligned}\right\} \tag{2.7}$$

Using (2.5) and (2.7) we have

$$\left.\begin{array}{l}
-k^2AB=\frac{B''}{2}-\frac{B'^2}{4B}-\frac{1}{4}\left(\frac{A'B'}{A}+\frac{\dot{A}\dot{B}}{C}-\frac{\hat{A}\hat{B}}{D}\right),\ k^2BC=\frac{\ddot{B}}{2}-\frac{1}{4}\left(\frac{\dot{B}^2}{B}+\frac{B'C'}{A}+\frac{\dot{B}\dot{C}}{C}+\frac{\hat{B}\hat{C}}{D}\right)\\
k^2AC=\frac{\ddot{A}-C''}{2}-\frac{1}{4}\left(\frac{\dot{A}^2}{A}-\frac{C'^2}{C}-\frac{A'C'}{A}+\frac{\dot{A}\dot{C}}{C}+\frac{\hat{A}\hat{C}}{D}\right)\\
k^2B^2=-B+\frac{1}{4}\left(\frac{B'^2}{A}-\frac{\dot{B}^2}{C}+\frac{\hat{B}^2}{D}\right),-k^2BD=\frac{\hat{\hat{B}}}{2}-\frac{1}{4}\left(\frac{\hat{B}^2}{B}-\frac{B'D'}{A}+\frac{\dot{B}\dot{D}}{C}+\frac{\hat{B}\hat{D}}{D}\right)\\
-k^2AD=\frac{\hat{\hat{A}}+D''}{2}-\frac{1}{4}\left(\frac{\hat{A}^2}{A}+\frac{D'^2}{D}+\frac{A'D'}{A}+\frac{\dot{A}\dot{D}}{C}+\frac{\hat{A}\hat{D}}{D}\right)\\
k^2CD=\frac{\ddot{D}-\hat{\hat{C}}}{2}+\frac{1}{4}\left(\frac{\hat{C}^2}{C}-\frac{\dot{D}^2}{D}-\frac{C'D'}{A}-\frac{\dot{C}\dot{D}}{C}+\frac{\hat{C}\hat{D}}{D}\right)
\end{array}\right\} \tag{2.8}$$

We obtain values of A, B, C and D satisfying (2.7) as

$$A=\left(1-r^2k^2\right)^{-1},\quad B=r^2,\quad C=\left(1-r^2k^2\right),\quad D=\left(1-r^2k^2\right) \tag{2.9}$$

Therefore (1.6) become

$$ds^2=-\left(1-r^2k^2\right)^{-1}dr^2-r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right)+\left(1-r^2k^2\right)dt^2-\left(1-r^2k^2\right)du^2$$

Thus the theorem is proved.

3. RICCI TENSOR $K_{ij}^{\cdot\cdot kl}$

THEOREM: The Ricci tensors $K_{ij}^{\cdot\cdot kl}$ of de-sitter space time V_5 in a Narrow sense with the metric (2.1) are

$$\left.\begin{array}{l}
\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=\alpha_{19}=-k^2\ ,\\
\alpha_7=\alpha_8=\alpha_9=\alpha_{10}=\alpha_{11}=\alpha_{12}=\alpha_{13}=\alpha_{14}=\alpha_{15}=\alpha_{16}=\alpha_{17}=\alpha_{18}=0
\end{array}\right\} \tag{3.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{19}$ are the mixed components of $K_{ij}^{\cdot\cdot kl}$ of the de-sitter space time V_5 in a Narrow sense.

PROOF: Consider $K_{ij}^{\cdot\cdot kl}=g^{ka}g^{lb}K_{ijab}$ (3.2)

Using the equations (2.3) and (2.6), the mixed components of $K_{ij}^{\cdot\cdot kl}$ are given by

$$\left.\begin{aligned}
&\alpha_1 = K_{12}^{\bullet\bullet 12} = K_{13}^{\bullet\bullet 13} = \frac{1}{AB}K_{1212} = -k^2, \quad \alpha_2 = K_{24}^{\bullet\bullet 24} = K_{34}^{\bullet\bullet 34} = -\frac{1}{BC}K_{2424} = -k^2,\\
&\alpha_3 = K_{14}^{\bullet\bullet 14} = -\frac{1}{AC}K_{1414} = -k^2, \quad \alpha_4 = K_{15}^{\bullet\bullet 15} = \frac{1}{AD}K_{1515} = -k^2\\
&\alpha_5 = K_{23}^{\bullet\bullet 23} = \frac{1}{B^2\sin^2\theta}K_{2323} = -k^2, \quad \alpha_6 = K_{25}^{\bullet\bullet 25} = K_{35}^{\bullet\bullet 35} = -\frac{1}{BD}K_{2525} = -k^2\\
&\alpha_7 = K_{12}^{\bullet\bullet 24} = K_{13}^{\bullet\bullet 34} = -\frac{1}{BC}K_{1224} = 0, \quad \alpha_8 = K_{24}^{\bullet\bullet 12} = K_{34}^{\bullet\bullet 13} = \frac{1}{AB}K_{1224} = 0\\
&\alpha_9 = K_{12}^{\bullet\bullet 25} = K_{13}^{\bullet\bullet 35} = \frac{1}{BD}K_{1225} = 0, \quad \alpha_{10} = K_{25}^{\bullet\bullet 12} = K_{35}^{\bullet\bullet 13} = \frac{1}{AB}K_{1225} = 0\\
&\alpha_{11} = K_{24}^{\bullet\bullet 25} = K_{34}^{\bullet\bullet 35} = \frac{1}{BD}K_{2425} = 0, \quad \alpha_{12} = K_{25}^{\bullet\bullet 24} = K_{35}^{\bullet\bullet 34} = -\frac{1}{BC}K_{2425} = 0\\
&\alpha_{13} = K_{14}^{\bullet\bullet 15} = \frac{1}{AD}K_{1415} = 0, \quad \alpha_{14} = K_{15}^{\bullet\bullet 15} = -\frac{1}{AC}K_{1415} = 0\\
&\alpha_{15} = K_{15}^{\bullet\bullet 45} = -\frac{1}{CD}K_{1545} = 0, \quad \alpha_{16} = K_{45}^{\bullet\bullet 15} = \frac{1}{AD}K_{1545} = 0\\
&\alpha_{17} = K_{14}^{\bullet\bullet 45} = -\frac{1}{CD}K_{1445} = 0, \quad \alpha_{18} = K_{45}^{\bullet\bullet 14} = -\frac{1}{AC}K_{1445} = 0,\\
&\qquad\qquad \alpha_{19} = K_{45}^{\bullet\bullet 45} = -\frac{1}{CD}K_{4545} - k^2
\end{aligned}\right\} \quad (3.3)$$

Thus, above equations prove the theorem.

4. PRINCIPAL INVARIANTS OF K_{pq}:

THEOREM: The ten principal invariants of K_{pq} for de-sitter space-time with line element (2.1) are $-k^2$.

PROOF: We consider curvature tensor is a symmetric tensor by taking

$$K_{ijkl} \equiv K_{pq} \text{ [where } p \equiv (i\,j)\,,\ q \equiv (k\,l)] \quad (4.1)$$

of a ten dimensional representation of five dimensional quantities for the line element (2.1).
The indices p, q . . . run from 1 to 10, where

$$\left.\begin{aligned}
&1 \equiv (12)\,,\ 2 \equiv (13), 3 \equiv (14),\ 4 \equiv (15),\ 5 \equiv (23),\\
&6 \equiv (24),\ 7 \equiv (25),\ 8 \equiv (34), 9 \equiv (35),\ 10 \equiv (45)
\end{aligned}\right\} \quad (4.2)$$

(we will represent 10 by 0, for rest of calculations)
The non-vanishing Ricci tensors $K_p^{\bullet q} = K_{ij}^{\bullet\bullet kl} = g^{qm}K_{pm}$
can be expressed in terms of $\alpha_1, \alpha_2, \ldots, \alpha_{19}$ (for ten dimensional representation of five dimensional quantities) are

$$\left.\begin{aligned}
&K_1^{\bullet 1} \equiv K_{12}^{\bullet\bullet 12} = K_2^{\bullet 2} \equiv K_{13}^{\bullet\bullet 13} = \alpha_1,\ K_3^{\bullet 3} \equiv K_{14}^{\bullet\bullet 14} = \alpha_3, K_4^{\bullet 4} \equiv K_{15}^{\bullet\bullet 15} = \alpha_4,\ K_5^{\bullet 5} \equiv K_{23}^{\bullet\bullet 23} = \alpha_5,\\
&K_6^{\bullet 6} \equiv K_{24}^{\bullet\bullet 24} = K_8^{\bullet 8} \equiv K_{34}^{\bullet\bullet 34} = \alpha_2,\ K_7^{\bullet 7} \equiv K_{25}^{\bullet\bullet 25} = K_9^{\bullet 9} \equiv K_{35}^{\bullet\bullet 35} = \alpha_6, K_0^{\bullet 0} \equiv K_{45}^{\bullet\bullet 45} = \alpha_{19}
\end{aligned}\right\} \quad (4.3)$$

Using equations (4.3), the ten principal invariants λ's of K_{pq} are given as solution from the equation

$$\det\left(K_p^{\bullet q} - \lambda\delta_p^q\right) = 0 \quad \text{where} \quad \delta_p^q = 1\,,\ p = q\,; \qquad = 0\,,\ p \ne q \quad (4.4)$$

i.e.

$$\begin{vmatrix} \alpha_1-\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1-\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3-\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_4-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_5-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_6-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_6-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{19}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\alpha_1-\lambda)^2(\alpha_3-\lambda)(\alpha_4-\lambda)(\alpha_5-\lambda)(\alpha_2-\lambda)^2(\alpha_6-\lambda)^2(\alpha_{19}-\lambda)=0$$

Using equation (3.1), we have

$$\left(-k^2-\lambda\right)^{10}=0$$

$$\Rightarrow \{\lambda\} \equiv \lambda'^{s} = \left(-k^2,-k^2,-k^2,-k^2,-k^2,-k^2,-k^2,-k^2,-k^2,-k^2\right) \tag{4.5}$$

Thus the theorem is proved.

Further if we take $k \to 0$ then (2.1) becomes the line element of the Minkoski space-time in s.s. coordinate system in a Narrow sense of V_5 i.e.

$$ds^2 = -dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + dt^2 - du^2 \tag{4.6}$$

All of $\alpha'^{s}\left(\text{i.e.}\,\alpha_1, \alpha_2, \ldots, \alpha_{19}\right)$ become 0 and ten principal invariants λ'^{s} become 0.

5. CONCLUSION:

Using [3] we can state that, if all ten principal invariants λ'^{s} are equal to constant $-k^2(\neq 0)$ or 0, then the spherically symmetric space-time V_5 in Narrow sense is de-sitter space-time V_5 or Minkoski space-time V_5 respectively in Narrow sense. The spherically symmetric line element in a Canonical coordinate system is given respectively by (2.1) or (4.6) to within a transformation of t.

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