STABILITY ANALYSIS OF A TWO DIMENSIONAL NON LINEAR MAP

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Abstract
In this paper a two dimensional non linear map is taken which represents a predator-prey model. It has been discussed thoroughly the nature of the fixed points of the map and the stability of the future trajectory of the model. A suitable method is applied to get the control parameter values which give a desired stability of the trajectory either in one or both the principal directions of the phase space.

Key Words: Fixed points/Periodic points/Stability of fixed points

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1.1 Introduction:

A very simple discrete model of the interaction between predator P and prey N, where it is assumed that predator can consume prey without limit, is given by

\[ N_{t+1} = r \cdot N_t e^{-bP_t} \]
\[ P_{t+1} = N_t(1 - e^{-aP_t}) \]

Where a, b, r > 0. Murray[10,11] gave a special case of the above model taking a=b and it is noticed that the model is unrealistic in the sense that the solutions can grow unboundedly with t. Hone, Irle and Thurura[4] discussed some of the beautiful aspects of dynamical behavior of the above said model and they have suggested a more realistic model

\[ x_{n+1} = x_n e^{r\left(1-\frac{x_n}{k}\right)-by_n} \]
\[ y_{n+1} = x_n(1 - e^{-ay_n}) \] \hspace{1cm} (1.1.1)

where a, b, c, k, r are adjustable parameters.

It is realistic in the sense that it represents Ricker curve \( x_{n+1} = x_n e^{r\left(1-\frac{x_n}{k}\right)} \) when restricted to one dimension. In case a=b it was discussed by Beddington et.al[1] which may be called as density dependent model.

In this chapter, we want to explore some dynamical aspects of the two dimensional ecological map (1.1.1). In the section 1.2, stability analysis of various fixed points is done. The main intention of this section is to find a procedure to obtain the value of the parameters which generates the fixed points of desired stability.

1.2 Stability analysis of fixed points of the model at various parameter values:

Let, \( f(x,y) = (xe^{r\left(1-\frac{x}{k}\right)-by}, x(1 - e^{-ay})) \)

Let \( f_1 = xe^{r\left(1-\frac{x}{k}\right)-by}, g_1 = x(1 - e^{-ay}) \)
The Jacobian matrix of the map is given as follows

\[
J = \begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y}
\end{pmatrix} = \begin{pmatrix}
e^r(1-\frac{x}{k})-by(1 - \frac{rx}{k}) & -bx e^{r(1-\frac{x}{k})-by} \\1 - e^{-ay} & axe^{-ay}
\end{pmatrix}
\]

The system is dissipative if

\[
\left| e^r(1-\frac{x}{k})-by(1 - \frac{rx}{k}) - bx e^{r(1-\frac{x}{k})-by} \right| < 1
\]

i.e. if

\[
\begin{vmatrix}
1 - \frac{rx}{k} & -bx \\
y & ax(1 - \frac{y}{x})
\end{vmatrix} < 1
\]

i.e. if

\[
\frac{a(k-rx)(x-y)+bky}{k} < 1
\]

In this context, we also wish to point out that the stability theory is intimately connected with the Jacobian matrix of the map, and that the trace of the Jacobian matrix is the sum of its eigenvalues and the product of the eigenvalues equal to the Jacobian determinant. If both of the eigenvalues of the above Jacobian matrix at some fixed point are less than one in modulus, then the fixed point attracts its nearby points from its basin of attraction and hence is said to be stable fixed point. On inspection it can be seen that (0,0) and (k,0) are two fixed points of the model satisfying the equation

\[
f(x,y) = (x,y) = (xe^{r(1-\frac{x}{k})-by}, x(1 - e^{-ay}))
\]

i.e.

\[
x = xe^{r(1-\frac{x}{k})-by}
\]

\[
y = x(1 - e^{-ay}) \quad (1.2.1)
\]

The eigenvalues of the Jacobian matrix J at the fixed point (0,0) are 0 and e^r. If r > 0, then e^r > 1 indicating (0,0) as unstable fixed point. Again for the fixed point (k, 0) the eigen values are 1-r and a k. If a k>1 then the fixed point (k, 0) already becomes an unstable fixed saddle point. Otherwise for a k <1, fixed point (k,0) is stable till
\[ |1-r|<1 \]

i.e. \(-1<1-r<1\).

i.e. \(-2<-r<0\).

i.e. \(0<r<2\).

So, we see that \(r=2\) is the parameter value where the fixed point \((k, 0)\) loses its stability provided a \(k<1\). In other words we may infer that for a \(k<1\) and if the Jacobian \(J\) gives one of the eigenvalues greater than one in modulus for all fixed points except \((k, 0)\) of the map \((1.2.1)\), then \(r=2\) is the first bifurcation point of that class.

Now we investigate the existence of other fixed points say \((x, y)\) and their stability,

The characteristic polynomial of the Jacobian matrix is given as follows:

\[ \lambda^2 - \lambda X + Y = 0, \]  
where

\[ X = a(x - y) + 1 - \frac{rx}{k} \]

\[ Y = (a - b)x(1 - \frac{y}{x}) + bx - \frac{ra}{k} x^2 (1 - \frac{y}{x}) \]

i.e. \(Y = (a - b)(x - y) + bx - \frac{ra}{k} (x^2 - xy)\)

The eigen values of the Jacobian matrix for any fixed point \((x, y)\) is given by

\[ \lambda_1 \]

\[ = \frac{k + akx - rx - aky - \sqrt{((-k - akx + rx + aky)^2 - 4k(akx - arx^2 - aky + bky + arxy))}}{2k} \]

\[ , \]

\[ \lambda_2 \]

\[ = \frac{k + akx - rx - aky + \sqrt{((-k - akx + rx + aky)^2 - 4k(akx - arx^2 - aky + bky + arxy))}}{2k} \]

Let ,
\[ k + ax - rx - aky \pm \sqrt{\{(k - akx + rx + aky)^2 - 4k(ax - arx^2 - aky + bky + arxy)\}} \]

\[ \pm \sqrt{\{(k - akx + rx + aky)^2 - 4k(ax - arx^2 - aky + bky + arxy)\}} \]

\[ = 2k(-1 - \varepsilon) - (k + akx - rx - aky) \]

Assuming \( 2k(-1 - \varepsilon) - (k + akx - rx - aky) > 0 \)

We take

\[ \sqrt{\{(k - akx + rx + aky)^2 - 4k(ax - arx^2 - aky + bky + arxy)\}} = 2k(-1 - \varepsilon) - (k + akx - rx - aky) \quad (1.2.2) \]

If \( 2k(-1 - \varepsilon) - (k + akx - rx - aky) < 0 \)

We take,

\[ -\sqrt{\{(k - akx + rx + aky)^2 - 4k(ax - arx^2 - aky + bky + arxy)\}} = \]

\[ 2k(-1 - \varepsilon) - (k + akx - rx - aky) \quad (1.2.3) \]

But both the equations (1.2.2) and (1.2.3) after solving and putting \( y = \frac{r}{b}(1 - \frac{x}{k}) \) (obtained from equation (1.2.1) and considering \( x \neq 0 \)), We have,

\[ b = \frac{arx(2 + \varepsilon - 2x - \frac{ak}{k} - \frac{ar}{k} - \frac{rx^2}{k})}{2 + 3\varepsilon + \varepsilon^2 + r + 2ax + aex - \frac{2rx}{k} - \frac{arx}{k} - \frac{arx^2}{k}} \quad (1.2.4) \]

Again if we put the value of \( b \) in the expression of \( \lambda_1, \lambda_2 \), we see that the eigenvalues are

\[ \lambda_1 = -\frac{rx}{k} + \frac{k(2 + \varepsilon - r)}{(2 + \varepsilon)k - rx} \quad \text{and} \quad \lambda_2 = -1 - \varepsilon. \]
If we choose \( \varepsilon, \lambda_2 \) will be controlled. Now if we want to find the parameters for which the model has fixed point \((x, y)\) other than \((k, 0)\) and \((0, 0)\) at which one of the eigenvalues is \(-1 - \varepsilon\), we consider the equations (1.2.1).

\[
i.e. \ x = xe^{r(1 - x/k)} - by
\]

\[
i.e. \ 1 = e^{r(1 - x/k)} \quad \text{[as} \ x \text{considered not equal to zero]}
\]

\[
i.e. \ y = \frac{r}{b}(1 - \frac{x}{k}) \quad (1.2.5)
\]

\[
\frac{r}{b}(1 - \frac{x}{k}) = x(1 - e^{-a_{\frac{r}{b}}(1 - \frac{x}{k})}) \quad (1.2.6)
\]

Where \( b \) satisfies (1.2.4).

For fixed value of \( a, k, r \) we may solve (1.2.6) for \( x \) if it exists. Then we put the value of \( x \) in the expression of (1.2.4) which will give the value of \( b \) and hence we calculate \( y = \frac{r}{b}(1 - \frac{x}{k}) \). Thus we get the fixed point for the model and the value of “\( b \)” will suffice that one of the eigenvalues is \(-1 - \varepsilon\).

Below in the table we give some of the examples using above procedure.

**Table 1.2.a** : Taking \( ak > 1(\varepsilon = -0.01)\)

<table>
<thead>
<tr>
<th>( a )</th>
<th>( k )</th>
<th>( r )</th>
<th>( b )</th>
<th>Fixed point</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>45.0</td>
<td>1.0</td>
<td>- 0.00736218</td>
<td>(67.2542, 67.1729)</td>
<td>(-0.99, 0.503598)</td>
</tr>
<tr>
<td>0.1</td>
<td>45.0</td>
<td>1.1</td>
<td>- 0.00705719</td>
<td>(63.2181, 63.1032)</td>
<td>(-0.99, 0.456158)</td>
</tr>
<tr>
<td>0.1</td>
<td>45.0</td>
<td>1.2</td>
<td>- 0.00663402</td>
<td>(59.8516, 59.6987)</td>
<td>(-0.99, 0.409248)</td>
</tr>
<tr>
<td>0.1</td>
<td>45.0</td>
<td>1.3</td>
<td>- 0.00610208</td>
<td>(56.9985, 56.8047)</td>
<td>(-0.99, 0.362827)</td>
</tr>
<tr>
<td>0.1</td>
<td>45.0</td>
<td>1.4</td>
<td>- 0.00546937</td>
<td>(54.5475, 54.3087)</td>
<td>(-0.99, 0.316854)</td>
</tr>
</tbody>
</table>
Here are some other examples

Table 1.2.b (taking $ak<1$) ($\epsilon = 0.01$)

<table>
<thead>
<tr>
<th>$a$</th>
<th>$k$</th>
<th>$r$</th>
<th>$B$</th>
<th>Fixed Point</th>
<th>Eigen values</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>1.0</td>
<td>-0.0117223</td>
<td>(93.5107,-13.2722)</td>
<td>(1.39598,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>1.1</td>
<td>0.0228227</td>
<td>(87.0049,-27.2238)</td>
<td>(1.43097,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>1.2</td>
<td>0.0140853</td>
<td>(81.4796,-39.6548)</td>
<td>(1.46279,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>1.3</td>
<td>0.0097147</td>
<td>(76.7241,-50.8414)</td>
<td>(1.49175,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>1.4</td>
<td>0.00701208</td>
<td>(72.585,-60.9919)</td>
<td>(1.51809,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>4.1</td>
<td>-0.00850896</td>
<td>(33.9667,-187.48)</td>
<td>(1.71973,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>4.2</td>
<td>-0.00881525</td>
<td>(33.4409,-189.885)</td>
<td>(1.71715,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>4.3</td>
<td>-0.00911807</td>
<td>(32.938,-192.215)</td>
<td>(1.71417,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>4.4</td>
<td>-0.00941768</td>
<td>(32.4565,-194.474)</td>
<td>(1.71081,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>4.5</td>
<td>-0.00971431</td>
<td>(31.995,-196.666)</td>
<td>(1.70708,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>4.6</td>
<td>-0.0100082</td>
<td>(31.5524,-198.793)</td>
<td>(1.70301,-1.01)</td>
</tr>
<tr>
<td>0.01</td>
<td>55.6</td>
<td>4.7</td>
<td>-0.0102994</td>
<td>(31.1273,-198.273)</td>
<td>(1.6986,-1.01)</td>
</tr>
</tbody>
</table>
In the next part we wish to control the other eigenvalue.

Let us consider $\lambda_1 = -1 - \hat{\epsilon}$

Then $-\frac{rx}{k} + \frac{k(2+\epsilon-r)}{(2+\epsilon)k-rx} = -1 - \hat{\epsilon}$

Solving for $x$ we get

$$x = \frac{1}{2r^2} \left(3kr + \epsilon kr + \hat{\epsilon}kr \pm kr\sqrt{(-7 - 2\epsilon + \epsilon^2 - 2\hat{\epsilon} - 2\epsilon \hat{\epsilon} + \hat{\epsilon}^2 + 4r)}\right)$$

Now we consider the equation (1.2.6) using (1.2.5)

$$\frac{r}{b} \left(1 - \frac{x}{k}\right) = x \left(1 - e^{-\frac{r}{b}(1 - \frac{x}{k})}\right)$$

If we put the value of $x$ and use the expression of $b$ of (1.2.4), we have

$$\frac{(3+\epsilon+\hat{\epsilon})(ak+r)\pm(ak-r)\sqrt{-7-2\epsilon+\epsilon^2-2\hat{\epsilon}^2-2\epsilon\hat{\epsilon}+\hat{\epsilon}^2+4r}}{2ar} = \frac{1}{2r^2} \left(3kr + \epsilon kr + \hat{\epsilon}kr \pm kr\sqrt{-7 - 2\epsilon + \epsilon^2 - 2\hat{\epsilon} - 2\epsilon \hat{\epsilon} + \hat{\epsilon}^2 + 4r}\right)$$

$$\left(1 - e^{-\frac{(3+\epsilon+\hat{\epsilon})(ak+r)\pm(ak-r)\sqrt{-7-2\epsilon+\epsilon^2-2\hat{\epsilon}^2-2\epsilon\hat{\epsilon}+\hat{\epsilon}^2+4r}}{2r}}\right)$$

i.e.
Taking a $k=t$, we have

$$\frac{(3+\varepsilon+\hat{\varepsilon})(ak+r)\pm(ak-r)\sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r}}{2r} = \frac{1}{2r^2} \left( 3akr + \varepsilon akr + \hat{\varepsilon} akr \pm \right.
\left. akr \sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r} \right)$$

$$\left( 1 - e^{-\frac{(3+\varepsilon+\hat{\varepsilon})(ak+r)\pm(ak-r)\sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r}}{2r}} \right)$$

Taking a $k=t$, we have

$$\frac{(3+\varepsilon+\hat{\varepsilon})(t+r)\pm(t-r)\sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r}}{2r} = \frac{1}{2r^2} \left( 3tr + \varepsilon tr + \hat{\varepsilon} tr \pm \right.
\left. tr \sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r} \right)$$

$$\left( 1 - e^{-\frac{(3+\varepsilon+\hat{\varepsilon})(t+r)\pm(t-r)\sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r}}{2r}} \right)$$

The equation (1.2.9) says that for a particular value of $\varepsilon$ and $\hat{\varepsilon}$, $r$, if the solution exists for $t$, $t$ will give infinite combinations of $a$, $k$ but a unique value of “b” such that the model (1.1.1) has fixed points which gives eigenvalues from the Jacobian matrix $J$ as $-1-\varepsilon$ and $-1-\hat{\varepsilon}$.

For example if $\varepsilon=-0.01$ and $\hat{\varepsilon}=-1.503598$ and $r=1$, we solve the equation

$$\frac{(3+\varepsilon+\hat{\varepsilon})(t+r)+(t-r)\sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r}}{2r}$$

$$= \frac{1}{2r^2} \left( 3tr + \varepsilon tr + \hat{\varepsilon} tr + tr \sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r} \right)$$

$$\left( 1 - e^{-\frac{(3+\varepsilon+\hat{\varepsilon})(t+r)+(t-r)\sqrt{-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}^2+4r}}{2r}} \right)$$

Using Mathematica the above equation is solved numerically and the value of $t=4.5$.

For the fixed points we use the expression

$$\frac{1}{2r^2} (3kr + \varepsilon kr + \hat{\varepsilon} kr + kr \sqrt{(-7-2\varepsilon+\varepsilon^2-2\hat{\varepsilon}-2\hat{\varepsilon}+\hat{\varepsilon}^2+4r}))$$
Which gives $x = 67.2542$, $y = 67.1728$ hence $b = -0.00736218$ now if the Jacobian is calculated at this fixed point the eigenvalues are $-0.99$, $0.503598$ which verifies the theory.

References: