Groups of Automorphisms of Some Graphs

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Abstract
Many properties of graphs and their behavior can be studied much easier with Group Theory applications. Symmetric groups have been playing the basic role in the development of Graph Theory. Many complicated proofs in graph theory can be resolved by using very basic properties of groups. One of the areas in which group theory has been implemented is the symmetric groups, which are the key tools to study the graph theory and geometry. On the other hand many well known abstract concepts, such as Klein 4-group, Cyclic groups, Permutation groups have their applications in graph theory.

This paper deals with the automorphisms of some basic graphs, tackling some classic procedures and giving alternative approaches to reach conclusions and finally constructing the Peterson Graph and its automorphisms and relating this to symmetric groups.

Introduction
Section one in this paper introduces some basic concepts and definitions of graphs, which are covered by Wilson [5]. The necessary definitions of groups, symmetric groups, automorphisms of groups are available in many Abstract Algebra references. Biggs [1] introduced the relationship between graph theory and Algebra.

Section Two covers the properties of symmetric groups which will be used in the later sections.

Section Three deals with the automorphisms of graphs and their applications of groups. The relationship between the symmetric groups and the graph automorphisms are studied in this section.

Section Four focuses on the Peterson graph and its automorphisms, starting with the way of constructing this graph in terms of 2-subsets of a set of five elements. Proofs of some properties are also given in this section. Finally the proof of isomorphism of the automorphism of Peterson group to $S_5$ completes this section.
1. Basic Graphs

**Definition 1.1:** A graph \( G = (V, E) \) is a set of object \( V = \{v_1, \ldots\} \) called vertices and another set \( E = \{e_1, \ldots\} \) called the edges of \( G \) such that each edge \( e \) is defined by \((v_i, v_j)\) of the vertices.

Moreover, there exist two maps \( i: E \rightarrow V \) and \( t: E \rightarrow V \) such that \( i(e) \) is called the initial vertex of \( G \) and \( t(e) \) is called the terminal vertex of \( G \) for \( e \in E \).

The inverse of an edge \( e \) denoted by \( \bar{e} \) defined by

\[
i(\bar{e}) = t(e) \quad \text{and} \quad (\bar{e}) = i(e), \quad \text{so} \quad \bar{e} = e.
\]

Two vertices \( v_1 \) and \( v_2 \) are called adjacent if they have an edge \((v_1, v_2)\).

**Definition 1.2:** A loop at a vertex \( v \) is defined as an edge \( e \) where \((e) = t(e) = v\).

**Definition 1.3:** A graph \( G = (V, E) \) is called finite if the set \( V \) is finite.

**Definition 1.4:** The degree \( d(v) \) of a vertex \( v \) in a graph \( G = (V, E) \) is defined as the number of edges at \( v \).

In figure 1 shows that:

\[
d(v_1) = 4 \quad \text{(considering} \ e_1, \bar{e}_1, e_2, e_3)\]

Whereas \( d(v_2) = 2, d(v_3) = 3 \) and

\[
d(v_4) = 1.
\]

**Theorem 1.1** In any finite graph the number of vertices of odd degree is even.

**Proof:** See [2]

**Definition 1.5:** A graph in which the vertices have the same degree is called regular.

**Definition 1.6:** A subgraph \( \hat{G} = (\hat{V}, \hat{E}) \) of a graph \( G = (V, E) \) is a graph where \( \hat{V} \subseteq V \) and \( \hat{E} \subseteq E \).

If \( \hat{V} = V \) and \( \hat{E} \subseteq E \) then \( \hat{G} \) is called spanning subgraph, so the spanning subgraph contains all the vertices of a graph, but only some edges of the original graph.
Definition 1.7: Let $G = (V, E)$ and $\hat{G} = (\hat{V}, \hat{E})$ be graphs. A bijection $\phi: V \to \hat{V}$ is an isomorphism if $(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in \hat{E}$.

If $\hat{G} = G$ then $\phi$ is called an automorphism of $G$. Moreover, if $\phi: G \to \hat{G}$ is an isomorphism then $G$ isomorphic to $\hat{G}$ and we write $G \cong \hat{G}$.

2. Some Basic Groups

Definition 2.1 A set $G$ with an operation $*$ is called group if the following are satisfied:

1) For all $a, b \in G$, $a * b \in G$.
2) For all $a, b, c \in G$, then $a * (b * c) = (a * b) * c$.
3) There exists an element $e$, called identity so that $a * e = e * a = a$ for all $a \in G$.
4) For each $a \in G$, there exists an element $b \in G$ such that $a * b = b * a = e$, called the inverse of $a$ written by $a^{-1}$.

A group $G$ is called finite if it contains finite number of elements, and called cyclic if:

$G = \{a^n : a \in G \text{ and } n \in \mathbb{Z}\}$.

A typical example of finite cyclic group would be $\mathbb{Z}_n$.

Many groups are closely connected to geometrical shapes such as polygons of $n$ sides, including triangle.

Symmetric groups

Definition 2.2 A permutation of a set $S$ is a one-to-one, onto mapping from $S$ to itself.

Example Let $S = \{1, 2, 3\}$, the permutations on $S$ are given by

$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ and $d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

The set $\{e, a, b, c, d, f\}$ forms a group under composition called the symmetric group $S_3$. $S_n$ contains $n!$ elements.

The elements of $S_3$ can be expressed by:

$e = (1)(2)(3)$, $a = (1)(2\ 3)$, $b = (2)(1\ 3)$, $c = (1\ 2)(3)$, $d = (1\ 3\ 2)$ where $a^2 = b^2 = c^2 = 1$ and $d^3 = f^3 = 1 = a = a^{-1}$, $b = b^{-1}$, $c^{-1} = c$, $d^{-1} = (3\ 2\ 1) = f$ and $f^{-1} = (2\ 3\ 1) = d$.

A permutation $\pi \in S_n$ which interchanges two elements and fixed all the others is called a transposition.

Another example of a typical example of a group is $D_n$. This is the set symmetries of a polygon of $n$ sides. It is called the Dihedral group of on $n$ elements.

Definition 2.3 A Permutation $\pi$ in $S_n$ is called a cycle if it has at most one orbit containing more than one element. The length of a cycle is determined by the length of its largest orbit.

Eg: $\pi = (1)(2, 3)(4)$ is a cycle of length 2.

Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{pmatrix}$
Then \( \pi \) is split into the following orbits: \( \{1, 2, 3\}, \{3, 5\}, \{6\} \). Therefore, a transposition is a cycle of length 2.

A permutation of a finite set is even or odd according to whether it can be expressed on a product of an even number of transpositions or the product of an odd number of transpositions.

**Example** Let \( (1,4,5,6)(2,1,5) = (1,6)(1,5)(1,4)(2,5)(2,1) \)

This is an odd permutation.

**Dihedral Group** \( (D_n) \) This is the set of all symmetries of a polygon of \( n \) sides.

We consider \( D_5 \) which consists of all the symmetries of the pentagon. The symmetries are:

1. Five Rotations, each of \( 72^\circ \). Therefore, a cyclic group \( R \) of order \( 5 \cdot \{R_0, R_1, R_2, R_3, R_4\} \).
2. Five reflections (flips) at each vertex. These flips lack forming a group when composing any two flips and the non-existence of the identity element.

   However each flip \( f \) together with the identity \( R_0 \), i.e. \( F = \{R_0, f\} \) forms a group.

   Then \( D_5 = R \times F \) and \( |D_5| = |R||F| = 5 \cdot 2 = 10 \).

**Definition 2.4** The direct product of the groups \( G_1, ..., G_n \) when \( n \) is finite is defined by:

\[
G_1 \times ... \times G_n \text{ consisting elements of the form } (g_1, ..., g_n) \text{ where } g_i \in G_i, 1 \leq i \leq n.
\]

**Definition 2.5** A homomorphism from a group \( G \) to a group \( \hat{G} \) is a mapping \( f: G \rightarrow \hat{G} \) where \( f(g_1g_2) = f(g_1)f(g_2) \) where \( g_1, g_2 \in G \).

**Definition 2.6** An isomorphism is a one-to-one onto homomorphism.

**Definition 2.7** An automorphism of a group \( G \) is an isomorphism of a group \( G \) onto itself. The set of all automorphisms of a group \( G \) is denoted by \( \text{Aut}(G) \).

**Theorem 2.1** Let \( G \) be a group and \( \text{Aut}(G) \) be the set of all automorphisms of \( G \). Then \( \text{Aut}(G) \) forms a group under the compositions of functions.

**Proof** See [3]

3. **Groups of Graphs**

**Definition 3.1** Let \( G = (V, E) \) be a finite graph. An automorphism of \( G \) is a permutation of the vertex set that satisfies the condition \( \{u_i, u_j\} \in E(G) \) if and only if \( \{\Phi(u_i), \Phi(u_j)\} \in E(G) \).

The automorphism group of \( G \) is the set of permutations of the vertex set that preserve adjacency. See [4]

\[
\text{Aut}(G) = \{\pi \in \text{Sym}(V): \pi(E) = E\}.
\]

**Theorem 3.1** The set \( \text{Aut}(G) \) of all automorphisms of a group \( G \) forms a group under compositions of functions.

**Proof** See Road [3].

**Definition 3.2** An edge automorphism on a graph \( G = (V, E) \) is a permutation of the set of edges of \( G \) satisfying \( e_i, e_j \) are adjacent if and only if \( \Phi(e_i), \Phi(e_j) \) are also adjacent.
Theorem 3.2 \( \text{Aut}(K_n) \cong S_n \) (isomorphic).

Proof \( K_n \) contains \( n \) vertices which are all connected to each other. Each vertex is connected to \( n - 1 \) edges. Each vertex from \( K_n \) is mapped to another vertex. The other \( n - 1 \) vertices are connected to \( n - 2 \) vertices and so on.

Therefore, \( \text{Aut}(K_n) \) contains \( n(n - 1)(n - 2) \ldots 2 \times 1 \) elements and this is given by \( n! \). Since \( S_n \) contains \( n! \) elements.

Therefore, each element in \( \text{Aut}(K_n) \) is mapped to an element in \( S_n \). Therefore, \( \text{Aut}(K_n) \cong S_n \).

Klein four-group \( V_4 \)

Klein four-group \( V_4 \) is a group which is equivalent to direct sum of \( \mathbb{Z}_2 \) by \( \mathbb{Z}_2 \) ie \( V_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). It is a group of order 4 and each element is of order 2, which can be represented by \( (a, b | a^2 = b^2 = (ab)^2 = 1) \).

\( V_4 \) can also be represented by \( \{00, 01, 10, 11\} \). \( V_4 \) has a Cayley table as follows:

<table>
<thead>
<tr>
<th>*</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>ab</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>ab</td>
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<tr>
<td>a</td>
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<td>1</td>
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<td>b</td>
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<td>ab</td>
<td>1</td>
<td>a</td>
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<tr>
<td>ab</td>
<td>ab</td>
<td>b</td>
<td>a</td>
<td>1</td>
</tr>
</tbody>
</table>

\( V_4 \) can also be represented by \( \{1, 3, 5, 7\} \) with multiplication modulo 8. It is the symmetry group of Rhombus is the vertical reflection, the horizontal reflection and half a circle rotation.

Example of Automorphisms of graphs:

Let \( V = \{a, b, c, d\} \). Form the 2-elements subsets of \( V \) as follows : \( \{ab, ac, ad, bc, bd, cd\} \)

Take edge set as: \( E = \{ab, ac, ad, bc, bd, cd\} \)

Then we get the graph \( G = (V, E) \) of order \( |V| = 4 \). Look at figure 3.

Define the automorphism \( \varnothing_i \) as following \( \varnothing_0 : G \rightarrow G , \varnothing_0 = \text{id} \).

\( \varnothing_i \circ \varnothing_j \) is defined by the transformation acted by \( \varnothing_j \) followed the action of \( \varnothing_i \). This will give another transformation within \( \varnothing \).

\( \varnothing_0 = e \) \( \varnothing_1 \) Equivalent to vertical reflection \( \varnothing_2 \) Equivalent to horizontal reflection \( \varnothing_3 \) 180° rotation

Figure 4

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\((\emptyset_1)^2 = 1 \quad (\emptyset_2)^2 = 1 \quad \emptyset_3 = \emptyset_1 \emptyset_2 \text{and} (\emptyset_3)^2 = 1\)

\[\emptyset_0 \to 1\]

\[\emptyset_1 \to a \quad a^2 = 1\]

\[\emptyset_2 \to b \quad b^2 = 1\]

\[\emptyset_3 \to ab \quad (ab)^2 = 1\]

\[\emptyset \text{ Forms Klein 4-groups}\]

4. Automorphisms of Peterson Graph

Peterson Graph

To construct the Peterson graph, draw a cycle of length 5. At each vertex draw one additional edge to a new vertex. Draw two new edges from the new vertex producing a star in the centre. The graph is called Peterson graph which is 3-regular graph.

\[\text{Figure 5}\]

Another method to construct Peterson graph

Take \(A = \{0, 1, 2, 3, 4\}\), \(V = [A]^2\). Construct the vertices of Peterson graph as the two subsets of \(A\). ie \(\{0,1\}, \{0,2\}, \ldots, \{4,1\}, \{2,3\}, \ldots\)

By using the formula \(\binom{5}{2} = 10\), \(A\) will give exactly 10 vertices.

The edges are the disjoint 2-subsets

\[\text{Figure 6}\]
Lemma 4.1  Peterson graph contains no 3-cycles nor 4-cycle.

Proof: Suppose there exists a 3-cycle.

This can be represented as in figure (7).

Four elements are needed to construct vertices $v_1$ and $v_2$.

To construct $v_3$ an additional (sixth) element is needed (say $f$) and this is impossible.

For 4-cycle, the two possibilities are given in figure (8) and figure (9), showing that this is not possible to join to $v_4$ as they must have common elements namely $b$ in Figure (9) and $a$ in Figure (8).

Lemma 4.2  The symmetric group $S_5$ on $A$ acts as a group of automorphism on $P$.

Proof: Now let $A = \{0, 1, 2, 3, 4\}$. Any permutation $g$ of $A$, permutes $|A|^2$.

Also $|\{(a, b, c, d)\}| = \{|(g(a), g(b), g(c), g(d))\}|$ for all $a, b, c, d \in A$.

$\therefore \{a, b\}$ is disjoint to $\{c, d\}$ if and only if $g((a, b))$ is adjacent to $g((c, d))$.

Finally we introduce a different argument to prove the following theorem.

Theorem 4.1  The automorphism group of Peterson graph is isomorphic to the symmetric group $S_5$.

Proof: Let $P$ be the Peterson graph.

Let the function $f: P \rightarrow P$ be an automorphism mapping $\{a, b\}$ to a vertex $v$.

$Aut(P)$ is $S_5$ transitive on the vertices of $P$.

Therefore, if $v$ and $\{a, b\}$ are vertices in , then there is some $g \in S_5$ such that $g_1(v) = \{a, b\}$.

The automorphism $g_1 \circ f$ fixes $\{a, b\}$ and permutes the three adjacent edges of $\{a, b\}$.

The elements in $S_5$ which fix $\{a, b\}$, is a group generated by the permutation $(a, b), (c, d, e)$ and $(c, d)$ permutes the three adjacent edges of $\{a, b\}$ in all possible ways. This is the subgroup in $Aut(a, b)$ generated by $(c, d, e)$ and $(c, d)$ acts like $Sym(\{c, d\}, \{c, e\}, \{d, e\})$.

Now choose a permutation $g_2$ in $Aut(a, b)$ such that $g_2 \circ g_1 \circ f$ not only fixes $\{a, b\}$ but also each of three adjacent edges.

Continue in this way until $g_r \circ \ldots \circ g_2 \circ g_1 \circ f$ fixes all vertices of $P$.

So $g_r \circ \ldots \circ g_1 \circ f$ will be the identity automorphism. This means that $f = g_1^{-1} \circ g_2^{-1} \circ \ldots \circ g_r^{-1}$ as each $g_i \in Aut(P)$.
References


