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FRACTAL: A SET WHICH IS LARGER THAN THE UNIVERSE

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ABSTRACT

Fractal is a set, which geometric pattern is self-similar at different scales. It has a fractal dimension, which strictly exceeds its topological dimension. It can be a non-integer. We calculate fractal dimension for only exact self-similar fractals. We show a strong link between chaos theory and fractals in an informal manner and study about space-filling curves, which are special type of fractals having fractal dimension two. Fractals need not only to be self- similar in nature. We explain about different types of fractals depending upon their geometric patterns. Finally we discuss about the Mandelbrot set, one of the most complex structure in mathematics and the Julia set. The Julia set can be connected or disconnected. The connectedness of the Julia set depends on the structure of the Mandelbrot set.

1. Introduction

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." – Benoˆıt Mandelbrot

It is hard to fathom how extraordinary beautiful patterns like cloud formations, the veins of leaves and the branching of trees are so ubiquitous in nature. Surprisingly, these naturally occurring self-similar intricate patterns have a connection to mathematics. Benoˆıt Mandelbrot, professor at Yale University, first studied these self-similar patterns in nature in a graphical and structured manner in 1970s. He called these patterns *fractals*. Though before him many other mathematicians like Georg Cantor, Gaston Julia, Henri Poincaŕ e and Helge Von Koch studied these patterns, their analysis was limited to what they could depict in manual drawings.

To understand fractals, let us first ask ourselves the following question: How long is the coast of India? To measure this, take a fixed finite length *x* and move along the coastline in steps of length *x* until we reach or cross the end of the coastline. Multiplying the number of steps we took before we reached the end with *x*, we obtain a very crude approximation *L(x)* for the length of the coastline. It is true that by reducing *x*, we could possibly improve the approximation, but no matter how hard we try to better our estimate, the infinite irregularities in the coastline would render this endeavor rather futile. That means the length of the coastline is infinite having finite area [4]! In other words, the coastline is a fractal.

What is a fractal? A fractal is a spatial object whose geometric characteristics include *irregularity*, *scale-independence* and *self-similarity* [4]. Self-similarity means the fractal repeats itself at different scales (dilation?) [4]. Generally people have the idea that a continuous smooth curve has a unique tangent at every point on it except at the corners but most fractals are counterintuitive as these are continuous and nowhere differentiable curves due to their irregularity. Therefore, fractals are pathological objects, that is, they are very counterintuitive or badly behaved [4].

2. Fractal dimension

Formally, a fractal is a mathematical set having a *fractal dimension*, which strictly exceeds its *topological dimension* [1]. The fractal dimension is a number associated with every fractal that measures the roughness of the fractal [4]. Informally, the number of independent directions occupied by an object in space is the dimension of that object. For example, a line is 1-dimensional, a square is 2-dimensional and a cube is 3-dimensional. The motion of any body in phase-space requires three position coordinates and three momentum coordinates. Our finger can be bent by many independent angles and it requires 10 coordinates to describe its state perfectly [4]. In the pure mathematical subject of topology, many objects can be continuously deformed to one another but will continue to have the same dimension [3]. This type of dimension is called topological dimension [4]. For a better understanding of the fractal dimension, we need to know a little bit more than just its topological properties. So the fractal dimension is defined in a new way taking distance (scaling properties) or metric properties into consideration.

Naively, a *metric space* is a set where distance between elements of the set is defined [2]. Any subset of the metric space (*H(X),h*) is a fractal. Here h is the Hausdorff metric, (*X,d'*) is a complete metric space and points of *H(X)* space are the compact subsets of X, a non-empty set [1]. (There is a difference between any non-empty compact subset of *X* and any subset of *H(X)*.) [1]. Hausdorff dimension is a non-negative real number associated with every metric space. Felix Hausdorff first introduced it in 1918.

Fractals can have non-integer dimensions. Fractal dimension can be calculated by the formula $d = log(N)/log(n)$ [4]. Where linear size of that object can be reduced by $1/n$ in each spatial direction and N is the increase of the measure (length, area, volume) by n^d times from its original value and d is the fractal dimension. For example, the Koch curve has fractal dimension $d=log(4)/log(3)= 1.26$, the Cantor set has $d=log(3)/log(2)=0.6309$. But this definition of dimension is restricted only to self-similar fractals i.e., it must coincide with small piece of itself when suitably magnified. Hausedorff dimension can be used as more generalized way for calculating fractal dimension.

3. Chaos theory and fractals

The study of fractals becomes more interesting when we study *chaos theory*. Chaos theory is the study of unstable aperiodic behavior on nonlinear *dynamical systems* like weather systems [2]. Naively, a dynamical system is any system that changes dynamically with time. Nonlinear (deterministic chaos) dynamical system means that the output of the system does not depend directly on the input and any changes occurring at inputs does not affect the variables or the output at the same time [2]. Chaos deals with dissipative (energy loss that happens as a function of time) nonlinear dynamical systems and can be represented by nonlinear differential equations. Edward Lorenz, a meteorologist, first discovered chaos theory when he described the butterfly effect in 1961. Butterfly effect describes about the sensitive dependence of a system on initial conditions, which is one of the cause for the chaotic motion. An attractor is a set of stable conditions of a dynamical system [2]. An attractor can be chaotic (strange) or non-chaotic. A fractal can describe the geometric nature of an *attractor*. Chaotic attractors are the attractors with a fractal structure. This is how chaos theory and fractals are related.

4. Types of fractals

Fractals need not always be self-similar since they can also be *self-affine* in nature. Self-affine means that the ratio of self-similarity in different directions is different. Self-similarity can also be of several types such as exact self-similarity, quasi-self-similarity and statistical self-similarity. Exact self-similarity means that the fractal appears exactly identical, quasi self-similarity means that the fractal appears approximately identical and statistical self-similarity means that the fractal preserves its statistical similarity at different scales. The coast of India has statistical self-similarity and Koch Snowflake has exact self-similarity.

Another type of fractal is the *space-filling curve*. Space-filling curve is a continuous function with do-main [*0, 1*] and range is a 2-dimensional unit square (or more generally an *N*-dimensional hypercube). Intuitively, it is a continuous curve in 2 or 3 (or higher) dimensions and can be thought of as the path of a continuously moving point. For example the Peano curve which covers all the points in a unit square and snowflake sweep, which fills the Koch snowflake. Both the curves have fractal dimension two. There are other types of fractals like Base-Motif fractals, Julia set, IFS (Iterated Function System) fractals, plasma fractals, fractal canopies, paper-folding fractals, shape-replacement fractals etc. Several types like iteration, chaos game, fractal transformation; l-system, quaternion and strange attractors can construct these types of fractals.

5. Mandelbrot set and Julia set

Lets talk about one of the most complex structures of mathematics, *the Mandelbrot set*. It is a quasi- self-similarity type of fractal. It can be constructed by a formula iteration. The formula for the iteration is $F(z)=z^2+c$. The iteration is done in a complex plane, where the input is $z=0+i0$ (the critical point of F(z)) and c is any complex number [5]. If the output tends to zero (which means F(z) at $z=0+0i$ is bounded), the point c is contained within the set (the boundary of Mandelbrot set) and if the output tends to infinity, then c is outside the set [5].

The *Julia set* is the boundary of the *filled-in Julia set*. The filled-in Julia set is formed by a collection of points (*z*) for which *F(z)* do not tend to infinity after infinite iterations for an arbitrary c (the same *c* as in the Mandelbrot Set). The Julia set can be connected or disconnected [5]. Although both the sets are different, the interconnected nature of the Julia set depends on the Mandelbrot set in the following manner. The Mandelbrot set is the set of *c*, for which the Julia set is connected.

Conclusion

The theory of fractals can be applied to several fields like astronomy, biology, chemistry, art, music, weather, computer graphics, animation and data compression. It is now also being used for Euclidean tessellation.

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