FIXED POINT THEOREM USING WEAK COMPATIBILITY IN MENGES SPACE

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Abstract

In this paper, I present a fixed point theorem for six self mappings in Menger space under the condition of weak compatibility, which is generalize the result of Pant and Chauhan[2].

Keywords
Triangular norm, Menger space, common fixed point, weakly compatible mappings
1 Introduction

Menger[7] introduced the notion of probabilistic metric space, which is a generalization of metric space. The theory of probabilistic metric space has developed in many direction especially in nonlinear analysis and application. The idea of Menger was to use distribution function $F_{x,y}$ instead of nonnegative real numbers as value of metric. The important development of fixed point theory in Menger space was due to Sehgal and Bharucha –Reid [11]. Sessa[14] introduced weakly commuting maps in metric space. Jugnck [6] enlarged this concept to compatible maps. The notion of compatible maps in Menger space has been introduced by Mishra[11]. Singh and Jain [15] generalized the result of Mishra[11] using the concept of weak compatibility.

In this paper, I generalize the result of B.D Pant and Sunny Chouhan [2].

2 Preliminaries

Definition 2.1 A triangular norm $*$ (shorty t-norm) is a binary operation on the unit interval $[0, 1]$ such that for all $a, b, c, d \in [0, 1]$ the following conditions are satisfied:

(a) $a * 1 = a$
(b) $a * b = b * a$
(c) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(d) $a * (b * c) = (a * b) * c$.

Examples of t-norms are $a * b = \max\{a + b - 1, 0\}$ and $a * b = \min\{a, b\}$.

Definition 2.2 (Schweizer and Sklar [16]) The ordered pair $(X, F)$ is called a probabilistic metric space (shortly PM-space) if $X$ is a nonempty set and $F$ is a probabilistic distance satisfying the following conditions:

for all $x, y, z \in X$ and $t, s > 0$,

(FM-0) $F_{x,y}(t) = 1 \iff x = y$

(FM-1) $F_{x,y}(0) = 0$

(FM-2) $F_{x,y} = F_{y,x}$;

(FM-3) $F_{x,z}(t) = 1, F_{z,y}(s) = 1 \Rightarrow F_{x,y}(t + s) = 1$.

The ordered triple $(X, F, *)$ is called Menger space if $(X, F)$ is a PM-space,* is a t-norm and the following condition is also satisfies:

for all $x, y, z \in X$ and $t, s > 0$,

(FM-4) $F_{x,y}(t + s) \geq F_{x,z}(t) * F_{z,y}(s)$.

Definition 2.3 (Singh and Jain [15]) Self maps $A$ and $B$ of a Menger space $(X, F, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Lemma 2.4(Singh and Jain [15]) Let $\{x_n\}$ be a sequence in a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$. If there exists a constant $k \in (0, 1)$ such that

$F_{x_n,x_{n+1}}(kt) \geq F_{x_{n-1},x_n}(t)$

for all $t > 0$ and $n = 1, 2...$, then $\{x_n\}$ is a Cauchy sequence in $X$. 
Lemma 2.5 (Singh and Jain [15]) Let \((X, F, \ast)\) be a Menger space. If there exists \(k \in (0, 1)\) such that
\[F_{x,y}(kt) \geq F_{x,y}(t)\]
for all \(x, y \in X\) and \(t > 0\), then \(x = y\).

### 3 Main Result

**Theorem 3.1.** Let \(A, B, S, T, L\) and \(M\) be self mappings on a Menger space \((X, F, \ast)\) where \(\ast\) is the min \(t\)-norm and satisfying:

1. \(AB(X) \subseteq M(X)\) and \(ST(X) \subseteq L(X)\)
2. either \(AB(X)\) or \(M(X)\) or \(ST(X)\) or \(L(X)\) is a complete subspace of \(X\)
3. The pairs \(\{ST, M\}\) and \(\{AB, L\}\) are weakly compatible;
4. \(ST = TS\) and \(AB = BA\)
5. \(MT = TM\) or \(MS = SM\) and \(LA = AL\) or \(LB = BL\)
6. there is a \(k \in (0, 1)\) such that
\[
F_{\alpha y,ST v} (ku) \geq F_{y^{2n},y^{2n}} (u) * F_{ST v,ST v} (u) * F_{ST v,ST v} (u)
\]
for all \(x, y \in X\), for all \(u > 0\), for all \(\alpha \in (0, 2)\) and for some positive integer \(m\).

Then \(A, B, S, T, L\) and \(M\) have a unique common fixed point in \(X\).

**Proof:** Let \(x_0 \in X\). From the condition (3.1.1) \(\exists x_1, x_2 \in X\) such that
\[ABx_0 = Mx_1 = y_0\]
and \(STx_1 = Lx_2 = y_1\).

Inductively, we construct sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[ABx_{2n} = MX_{2n+1} = y_{2n}\]
and \(STx_{2n+1} = LX_{2n+2} = y_{2n+1}\) for \(n = 0, 1, 2, \ldots\).

Using result of lemma 2.4 we can be shown that \(\{y_n\}\) is a Cauchy sequence in \(X\) and so the subsequences \(\{y_n\}\) are also Cauchy in \(X\).

**Case I :** Suppose that either \(AB(X)\) or \(M(X)\) is a complete subspace of \(X\). Since \(\{y_{2n}\} \subseteq AB(X) \subseteq M(X)\), there is \(z \in X\) such that \(y_{2n} \rightarrow z\) as \(n \rightarrow \infty\) and \(y_{2n+1} \rightarrow z\) as \(n \rightarrow \infty\). Clearly, \(z \in M(X)\). So, there is a \(v \in X\) such that \(z = Mv\).

Now, taking \(x = x_{2n}\) (where \(n \geq 1\)), \(y = v\) and \(\alpha = 1\) in (3.1.6), we get that
\[
F_{\alpha y,ST v} (ku) \geq F_{y^{2n},y^{2n}} (u) * F_{ST v,ST v} (u) * F_{ST v,ST v} (u)
\]
for all \(u > 0\), \(\alpha \in (0, 2)\) and for some positive integer \(m\).

Since \(\{ST,M\}\) is weakly compatible, it follows that \(MSTv = STMv\)
\[\Rightarrow Mz = STz.\]
Again, taking \( x = x_{2n} \), \( y = Mz \), \( \alpha = 1 \) and using the fact that \( STz = Mz \) in (3.1.6), we get that
\[
F_{mz}(ku) \geq F_{mz}(u) * F_{z,Mz}(u) * F_{z,y_{2n}}(u) * F_{y_{2n},Mz}(u).
\]
Now, as \( n \rightarrow \infty \), we get that
\[
F_{mz}(ku) \geq F_{mz}(u) \text{ for all } u > 0.
\]
⇒ \( STz = Mz = z \).
Since \( ST = TS \), we have \( ST(Tz) = TS(Tz) = T(STz) = Tz \).
Now, taking \( x = x_{2n} (n \geq 1) \), \( y = Tz \) and \( \alpha = 1 \) in (3.1.6), we get that
\[
F_{mz}(ku) \geq F_{mz}(u) \text{ for all } u > 0.
\]
⇒ \( STz = Mz = z \).
Since \( ST = TS \), we have \( ST(Tz) = TS(Tz) = T(STz) = Tz \).
Now, taking \( x = x_{2n} (n \geq 1) \), \( y = Tz \) and \( \alpha = 1 \) in (3.1.6), we get that
\[
F_{mz}(ku) \geq F_{mz}(u) \text{ for all } u > 0.
\]
\[ L_z, L_z(u) \ast F_m \]
\[ y_{2n+1}, y_{2n}(u) \ast F_m \]
\[ L_z, y_{2n}(u) \ast F_{L_z, y_{2n}}(u) \ast F_{Y_{2n+1}, L_z}(u). \]

Now, as \( n \to \infty \) we get that, \( F_m \)
\[ L_z(z(u)) \geq F_m \]
\[ L_z, z(u), \text{ for all } u > 0. \]

Now, taking \( x = Az, y = x_{2n+1}, \alpha = 1 \) in (3.1.6) and using \( AB(Az) = Az \), we get that
\[ F_m \]
\[ A_z, y_{2n+1}(ku) \geq F_{A_z, y_{2n}}(u) \ast F_m \]
\[ y_{2n+1}, y_{2n}(u) \ast F_m \]
\[ A_z, y_{2n}(u) \ast F_{A_z, y_{2n}}(u) \ast F_{Y_{2n+1}, A_z}(u). \]

Now, as \( n \to \infty \) we get that \( F_m \)
\[ A_z, z(u) \geq F_m \]
\[ A_z, z(u), \text{ for all } u > 0. \Rightarrow Az = z. \]

Since \( AB = BA \), we have \( AB(Az) = A(BA)z = A(ABz) = Az \).

Suppose \( LA = AL \), so \( L(Az) = (LA)z = (AL)z = A(Lz) = Az \).

Hence, \( Az = Bz = Lz = Mz = Sz = Tz = z. \)

Similar is the case when \( LB = BL \). There we first show that \( Bz = z. \)

Case II: Suppose that either \( ST(X) \) or \( L(X) \) is a complete subspace of \( X. \)

we first get that \( Az = Bz = Lz = z \) and then \( Mz = Sz = Tz = z. \)

Hence, \( Az = Bz = Lz = Mz = Sz = Tz = z. \)

i.e, \( z \) is a common fixed point for \( A, B, S, T, L \) and \( M. \)

References