CRITICAL PATH ANALYSIS IN A PROJECT NETWORK USING RANKING METHOD IN INTUITIONISTIC FUZZY ENVIRONMENT

R. Sophia Porchelvi¹, G. Sudha²
¹Department of Mathematics, A.D.M College for women (Autonomous), Nagapattinam, India. E-mail: sophiaporchelvi@gmail.com
²Department of Mathematics, A.V.C. College (Autonomous), Mannampandal, India. E-mail: venkat_sudha@yahoo.in

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ABSTRACT

In this paper, an algorithm is presented to perform critical path analysis in an intuitionistic fuzzy environment. Triangular Intuitionistic fuzzy numbers are used to represent activity times in the project network. Ranking procedures are applied on intuitionistic fuzzy numbers to find the critical path. A numerical example illustrates the method.
1. Introduction
The critical path method is a vital tool for planning and control of complex projects. According to critical path method, the decision maker can control the time and cost of the project. CPM has been used in business management, factory, production etc. Here, a new approach for finding the critical path is introduced. The base idea behind this method is the use of fuzzy values under imprecise conditions.

The concept of fuzzy was introduced by Zadeh [19] in 1965. Various applications of fuzzy sets have been studied by researchers in different fields. T.J. Ross [12] published an interesting book on fuzzy sets theory and its applications in 2005. In this paper, a method for finding critical path in an Intuitionistic fuzzy project network is presented as discussed by [3]. A ranking procedure on Intuitionistic fuzzy numbers is applied for getting the critical paths (Refer [1]). This paper is organized as follows. In section 2, preliminary concepts and definitions in intuitionistic fuzzy set theory and the procedure for finding critical path using TIFN are provided. An algorithm is presented in section 3. An illustrative example to find the critical path is explained in section 4. The last section draws some concluding remarks.

2. Preliminary concepts
2.1 Intuitionistic Fuzzy Set: An intuitionistic fuzzy set $\tilde{A}$ in $X$ is given by a set of ordered triples: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x))| x \in X\}$, where $\mu_{\tilde{A}}, \nu_{\tilde{A}} : X \rightarrow [0,1]$ are functions such that $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$ for all $x \in X$. For each $x$ the numbers $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ represent the degree of membership and degree of non-membership of the element $x \in X$ to $A \subseteq X$, respectively.

2.2 Intuitionistic Fuzzy Number (IFN): An intuitionistic fuzzy subset $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x))| x \in X\}$ of real line $R$ is called an Intuitionistic Fuzzy Number (IFN) if the following holds:
(i) There exists $m \in R$, $\mu_{\tilde{A}}(m) = 1$ and $\nu_{\tilde{A}}(m) = 0$
(ii) $\mu_{\tilde{A}}$ is a continuous mapping from $R$ to the closed interval $[0,1]$ and for all $x \in R$, the relation $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$ holds.

The membership and non-membership function of $\tilde{A}$ is of the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & -\infty < x \leq m - \alpha \\ f_1(x), & x \in [m - \alpha, m) \\ 1, & x = m \\ h_1(x), & x \in [m, m + \beta] \\ 0, & m + \beta \leq x < \infty \end{cases}$$

where $f_1(x)$ and $h_1(x)$ are strictly increasing and decreasing function in $[m - \alpha, m]$ and $[m, m + \beta]$ respectively.

$$\nu_{\tilde{A}}(x) = \begin{cases} 1, & -\infty < x \leq m - \alpha^1 \\ f_2(x), & x \in [m - \alpha^1, m] \\ 0, & x = m \\ h_2(x), & x \in [m, m + \beta^1] \\ 0, & m + \beta^1 \leq x < \infty \end{cases}$$

Here $m$ is the mean value of $\tilde{A}$, $\alpha, \beta$ are called left and right spreads of membership function $\mu_{\tilde{A}}(x)$ respectively, $\alpha^1, \beta^1$ represents left and right spreads of non membership function $\nu_{\tilde{A}}(x)$ respectively.
2.3 Triangular Intuitionistic Fuzzy Number (TIFN)

A (TIFN) \( \tilde{A} \) is an intuitionistic fuzzy set in \( R \) with the following \( \mu_{\tilde{A}}(x) \) and \( \nu_{\tilde{A}}(x) \):

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
\frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\nu_{\tilde{A}}(x) = \begin{cases} 
\frac{a_2-x}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
\frac{x-a_2}{a_3-a_2}, & a_2 \leq x \leq a_3 \\
1, & \text{otherwise}
\end{cases}
\]

where \( a_1 \leq a_1 \leq a_2 \leq a_3 \) and \( \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1 \)

![Figure 1 Triangular Intuitionistic fuzzy number](image)

2.4 Chen and Cheng Metric Distance Ranking Procedure

Chen and Cheng proposed a metric distance method (see [1]) to rank fuzzy numbers. Let A and B be two fuzzy numbers defined as follows:

\[
g_A(x) = \begin{cases} 
g_A^L(x), & x < m_A \\
g_A^R(x), & x \geq m_A
\end{cases}
\]
where $m_A$ and $m_B$ are the mean of A and B. The metric distance between A and B can be calculated as follows:

$$D(A,B) = \sqrt{\int_0^1 (h_A^L(y) - h_B^L(y))^2 \, dy + \int_0^1 (h_A^R(y) - h_B^R(y))^2 \, dy}$$

where $h_A^L$, $h_A^R$, $h_B^L$ and $h_B^R$ are the inverse functions of $g_A^L$, $g_A^R$, $g_B^L$ and $g_B^R$ respectively. In order to rank fuzzy numbers, Chen and Cheng (see [1]) let the fuzzy number $B=0$ then the metric distance between A and 0 is calculated as follows:

$$D(A,0) = \sqrt{\int_0^1 (h_A^L(y))^2 \, dy + \int_0^1 (h_A^R(y))^2 \, dy}$$

, the larger value of $D(A,0)$ is the better ranking of A

According to Chen and Cheng, the membership function of A is defined as follows:

$$f_A(x) = \begin{cases} 
\frac{x-(\mu-\sigma)}{\sigma} & \text{if } \mu-\sigma \leq x \leq \mu \\
\frac{(\mu+\sigma)-x}{\sigma} & \text{if } \mu \leq x \leq \mu + \sigma 
\end{cases}$$

where $\mu$ and $\sigma$ are calculated as follows:

$$\sigma = \frac{a_2-a_1}{2}, \quad \mu = \frac{a_1+2a_2+a_3}{4}$$

where A becomes a TIFN, $A = (x, y, z, p, q, r)$. $a_1 = x + p$; $a_2 = y + q$; $a_3 = z + r$

The inverse functions $h_A^L$ and $h_A^R$ of $g_A^L$, $g_A^R$ respectively, are shown as follows:

$$h_A^L(y) = (\mu - \sigma) + \sigma y$$

$$h_A^R(y) = (\mu + \sigma) - \sigma y$$

### 3. Proposed Method for finding the critical path in intuitionistic fuzzy sense

**Notations:**

$N_j$ : The set of all nodes in a project network  
TIFES$_j$ : The triangular intuitionistic fuzzy earliest fuzzy time of j  
TIFLF$_j$ : The triangular intuitionistic fuzzy latest fuzzy time of j  
TIFTF$_{ij}$ : The triangular intuitionistic fuzzy total float of the activity i-j  
TIFT$_{ij}$ : The triangular intuitionistic fuzzy activity time of nodes i and j  
TIFP$_i$ : The triangular intuitionistic fuzzy $i^{th}$ path  
TIFP(j) : The set of all nodes connected to all predecessor activities of node j  
TIFS$_j$ : The set of all nodes connected to all successor activities of node j  
TIFCP: The triangular intuitionistic fuzzy completion time of path

**Algorithm:**

**Step 1**: Calculate $TIFES_j = \max\{TIFES_i \oplus TIFT_{ij} \mid i \in TIFP(j), j \neq 1, j \in N_j\}$ and $TIFES_1 = TIFL_1 = 0$
Step 2: Calculate TIFLF = \min \left\{ TIFL_F - TIFt_{ij} / j \in TIFS(j), j \neq n, j \in N_{ij} \right\} and TIFLF = TIFES_n.

Step 3: Calculate TIFT_F = TIFL_F \cap TIF T_{ij} ; i, j \in N_{ij}.

Step 4: Find all the possible paths and calculate TIFCP in a project network.

Step 5: Find the ranking value of TIFCP(P_i), i = 1,2,3,4 and compute the critical path.

4. Numerical Example

Consider a network with the triangular intuitionistic fuzzy arc lengths as shown below:

The possible paths of an triangular intuitionistic fuzzy project network are TIFP_1: 1-2-4-7-8 ; TIFP_2: 1-2-4-6-8 ; TIFP_3: 1-3-6-8 ; TIFP_4: 1-3-5-7-8.

Step 1

To calculate the triangular intuitionistic fuzzy earliest start:

Set TIFES_1 = (\langle 0,0,0 \rangle, \langle 0,0,0 \rangle)

Calculate TIFES_1, j = 2,3,4,5,6,7,8 by using

TIFES_1 = max \left\{ TIFES_i \oplus TIFt_{ij} / i \in TIFP(j), j \neq n, j \in TIF(N) \right\} and TIFES_1 = TIFLF = 0.

TIFES_2 = TIFES_1 \oplus TIFt_{12} = \langle (2,3,4), (5,6,7) \rangle, TIFES_3 = \langle (1,3,4), (4,5,6) \rangle,

TIFES_4 = \langle (3,6,9), (9,12,14) \rangle, TIFES_5 = \langle (2,5,7), (7,9,11) \rangle,

TIFES_6 = \langle (6,10,15), (14,19,22) \rangle, TIFES_7 = \langle (6,10,14), (15,19,22) \rangle,

TIFES_8 = \langle (9,16,22), (22,28,33) \rangle.
Step 2
To calculate the triangular intuitionistic fuzzy latest finish:
Set TIFLF_8 = (9, 16, 22), (22, 28, 33)
Calculate TIFLF_j = max {TIFLF_j \cap TIFTS(j), j \neq n, j \in TIF(N)} and TIFLF_n = TIFES_n

TIFLF_1 = (6, 10, 14), (15, 19, 22)

TIFLF_2 = (5, 6, 9), (10, 13, 15)

TIFLF_3 = (5, 6, 9), (10, 13, 15)

TIFLF_4 = (4, 7, 10), (10, 13, 16)

TIFLF_5 = (3, 4, 5), (6, 7, 9)

Step 3
To calculate the triangular intuitionistic fuzzy total float:
Calculate TIFTF_y with respect to each activity by using TIFTF_y = TIFLF_y \cup TIFES_y \cup TIFTF_y

TIFTF_12 = (1, 1, 1), (1, 1, 2)

TIFTF_13 = (1, 1, 1), (1, 1, 2)

TIFTF_14 = (1, 1, 1), (1, 1, 2)

TIFTF_15 = (1, 1, 1), (1, 1, 2)

TIFTF_16 = (0, 0, 0), (0, 0, 0)

TIFTF_17 = (1, 1, 1), (1, 1, 2)

TIFTF_18 = (1, 1, 1), (1, 1, 2)

Step 4
To get the possible paths as below:
Find all the possible paths and calculate TIFCP in a project network.
P = { (1, 2, 4, 7, 8), (1, 2, 4, 6, 8), (1, 3, 6, 8), (1, 3, 5, 7, 8) }

TIFP_1 = (1, 2, 4, 7, 8), then TIFCP(P_1) = TIFTF_12 \cup TIFTF_13 \cup TIFTF_14 \cup TIFTF_15 \cup TIFTF_16 \cup TIFTF_17 \cup TIFTF_18 = (2, 2, 2), (2, 2, 4)

TIFP_2 = (1, 2, 4, 6, 8), then TIFCP(P_2) = TIFTF_12 \cup TIFTF_24 \cup TIFTF_25 \cup TIFTF_68 = (4, 4, 4), (4, 4, 8)

TIFP_3 = (1, 3, 6, 8), then TIFCP(P_3) = TIFTF_13 \cup TIFTF_14 \cup TIFTF_16 \cup TIFTF_17 \cup TIFTF_18 = (9, 9, 11), (1, 1, 5, 20)

TIFP_4 = (1, 3, 5, 7, 8), then TIFCP(P_4) = TIFTF_13 \cup TIFTF_35 \cup TIFTF_57 \cup TIFTF_78 = (10, 5, 9), (1, 1, 5, 17)

Step 5
To obtain the critical path using ranking procedure:
P_i = (2, 2, 2), (2, 2, 4)

\( \sigma = 1, \mu = 4.5 \)

\( g^l_n(y) = 3.5 + y \) and \( g^h_n(y) = 5.5 - y \)

\( R(TIFCP(P_i)) = \frac{1}{2} \left[ \int_0^1 (3.5 + y)^2 dy + \int_0^1 (5.5 - y)^2 dy \right] = 6.4 \)

Similarly for finding R(TIFCP(P_j) = 12.8 ; R(TIFCP(P_k)) = 35.26 ; R(TIFCP(P_l)) = 31.67

Since R(TIFCP(P_1)) < R(TIFCP(P_2)) < R(TIFCP(P_3)) < R(TIFCP(P_4)),

Intuitionistic fuzzy critical path is 1-2-4-7-8
Conclusion

In this paper, a simple approach is provided to find the Triangular Intuitionistic fuzzy total duration times and the critical paths when the activity times are TIFNs. The proposed method gives triangular solutions and informations for making project management decisions.

References