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CORRECT ASSUMPTION APPROACH TO REFINED THEORY FOR ANALYSIS OF THICK BEAM

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Abstract

This paper presents correct assumption approach to refined theory for analysis of thick beam. To achieve this aim, the assumption that engineering vertical shear strain is not zero was fully implemented. For deformation involving the classical (elementary) deformation, $\gamma_{xz(c)}$ component and shear deformation, $\gamma_{xz(s)}$ component it was insisted that both are not equal to zero. That is $\gamma_{xz(s)} \neq 0 \; and \; \gamma_{xz(s)} \neq 0$. With these assumptions coupled with the assumption of uniform shear stress across the cross section of the beam, two new approaches, refined beam theories 2 and 3 (RBT2 and RBT3) were developed. Formulas for deflection and axial displacement coefficients were formulated. Also formulas for critical buckling load and fundamental natural frequency were formulated. These formulas were used in computing the values of center deflection, axial displacement, critical buckling load and fundamental natural frequency of a simply supported beam for span-depth (L/t) ratios of 100, 30, 20, 15, 10 and 4. It was discovered that RBT2 and RBT3 are giving exactly the same values. Their values were compared with values from higher order shear deformation theory designated RBT1 and they were very close to each other. The average percentage difference recorded from the comparison is 0.58%. This shows how close the values from RBT2/RBT3 are to values from RBT1.

Keywords:

shear deformation; vertical shear strain; deflection; axial displacement; buckling; frequency; span-depth ratio

Introduction

In classical beam theory (CBT), it is assumed that the vertical line, which is initially straight and normal to middle surface before bending remains straight and normal to the middle surface after bending. The implication being that vertical shear strain is zero. That is $\gamma_{xz} = \frac{du}{dz} + \frac{dw}{dx} = 0$. The integration of this expression results to expression of in-plane displacement of a classical beam, $u_c = -z\frac{dw}{dx}$ (Sayyad, 2011; Chikalthankar, 2013; Ibearugbulem et al., 2014). This assumption can be reasonable when the thickness of the beam is small (that is span-depth ratio is more than 20). When the span-depth ratio is less than 20, this assumption will no longer be reliable. This is the basis of first-order shear deformation theory (FSDT) introduced by Reissner (1942) and improved by Mindlin and Deresiewicz (1954). Because of the limitations (constant shear stress across the beam thickness) of this FSDT, the use of shear correction factor became necessary. To avoid the use of shear correction factor, second order shear deformation theory (SSDT) and subsequently higher-order shear deformation theories (HSDT) came up. The HSDT make use of different shear deformation functions (cubic polynomial, trigonometry, hyperbolic, exponential etc) called F(z) (Sayyad, 2001). For FSDT, this F(z) is equal to z. After critical study of the FSDT, SSDT and HSDT Ibearugbulem (2015) in his lecture note on advanced theory of elasticity revealed that a wrong assumption $\left(u_c = -z\frac{dw}{dx}\right)$ used in all of them. Since there exist shear stress across the section of the beam then engineering vertical shear strain will not be zero and we shall have $\gamma_{xz} = \frac{du}{dz} + \frac{dw}{dx}$. The integration of this expression shall result to $u = z\frac{dw}{dx}$. Note for classical beam, we have negative in expression for in-plane displacement, u while in refined beam theory (RBT) we should have positive. Another issue to note is that if the engineering vertical shear strain is not zero, then the complementary tensorial vertical shear strains must be equal to each other to maintain equilibrium. Thus, two of the complementary vertical shear strains are equal to the engineering vertical shear strain $\left(\gamma_{xz} = 2\frac{du}{dz} = 2\frac{dw}{dx}\right)$. Many previous scholars have erroneously applied this assumption in refined beam theories (Timoshenko, 1921; Hildebrand and Reissner, 1942 ; Kruszewski, 1949; Cowper, 1966 and 1968; Bickford, 1982, Krishna Murthy, 1984, Heyliger and Reddy, 1988; Bhimaraddi and Chandrashekhara, 1993, Ghugal and Shimpi, 2002; Ghugal, 2006; Ghugal and Sharma, 2009; Sayyad, 2011). They opined that total in-plane displacement (see figure 1) is made of classical beam displacement plus shear deformation displacement ($u = u_c + u_s$). From here, they introduce the classical beam displacement with negative sign into the expression for u $\left(u = -z\frac{dw}{dx} + u_s\right)$. The consequence of this assumption (introduction) is wrong expression for total engineering vertical shear strain

$$\gamma_{xz} = \left(-\frac{dw}{dx} + \frac{du_s}{dz}\right) + \frac{dw}{dx} \; instead \; of$$

$$\gamma_{xz} = \left(\frac{dw}{dx} + \frac{du_s}{dz}\right) + \frac{dw}{dx}.$$

Before we go ahead with the exposition, let us have a critical look on figure 1. From this figure and using trigonometric relations, the rotations or slopes (θ_c, θ_s and ϕ) can be defined for small angles as:

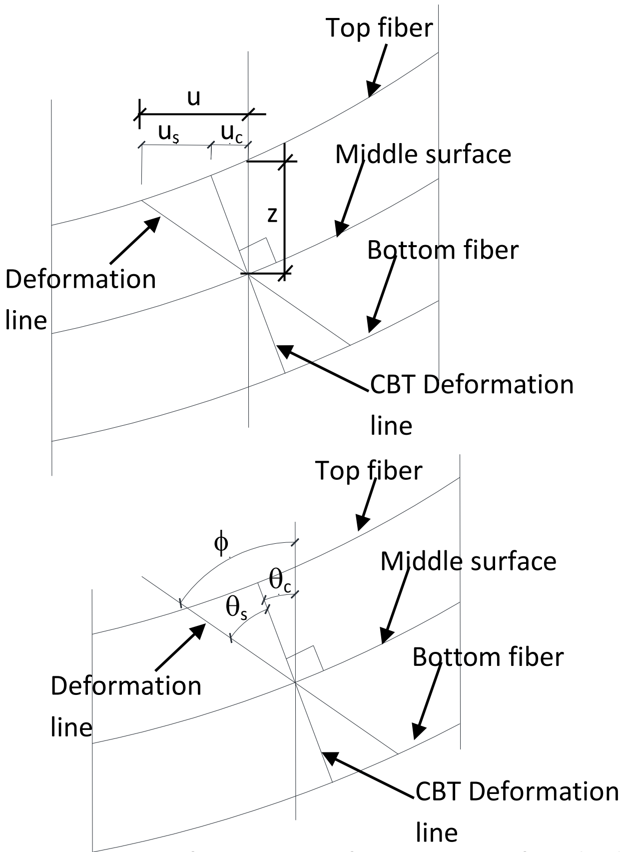

Figure 1: Deformation of a section of a thick beam

$\theta_c = \frac{u_c}{z}$; $\theta_s = \frac{u_s}{z}$; $\phi = \frac{u}{z}$. The consequence of this is that $u_c = z\theta_c$; $u_s = z\theta_s$; $u = z\phi$. It is common knowledge that slope in CBT is the first derivative of deflection, $\theta_c = \frac{dw}{dx}$. With this CBT displacement becomes: $u_c = z\frac{dw}{dx}$. It was based on these limitations that in his lecture note on advanced theory of elasticity, Ibearugbulem (2015) proposed and developed two new refined plate theories (RPT) for thick plates. Following his work, this paper is trying to formulate two new refined beam theories, RBT (with $\gamma_{xz} = \left(\frac{dw}{dx} + \frac{du_s}{dz}\right) + \frac{dw}{dx}$) in addition to the traditional RBTs (FSDT and HSDT), which assume engineering shear strain of $\gamma_{xz} = \left(-\frac{dw}{dx} + \frac{du_s}{dz}\right) + \frac{dw}{dx}$. The assumption in the two new RBTs is that F(z) is equal to z as in FSDT. Two different deflection equations, which are based on polynomial and trigonometric functions shall be used to study the pure bending, buckling and free vibration analyses of a simply supported beam. The traditional RBT (HSDT) shall be designated RBT1. One of the two new RBTs, that makes uses of $u = u_c + u_s = z\theta_c + z\theta_s$ shall be designated RBT2. The other new RBT, which uses $u = z\phi$ as a whole shall be designated RBT3.

Formulation of the theories

The formulation shall be displacement based. Direct variation of total potential energy approach shall also be adopted. The total potential energy of a beam is given as:

$$\Pi = b\int_x \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\sigma_x \varepsilon_x + \tau_{xz}\gamma_{xz}\right) dz\, dx - \int_x FF\, dx \qquad (1)$$

Where

$$FF = qw + \frac{N_x}{2}\left(\frac{d^2h}{dx^2}\right)^2 dx + \frac{\rho t \lambda^2}{2} w^2 \qquad (2)$$

Kinematic relations

Let define the deflection as:

$$w = A_1 h \qquad (3)$$

The classical in-plane displacement, u_c for RBT1 is given as:

$$u_c = -z\frac{dw}{dx} = -zA_1\frac{dh}{dx} \qquad (4)$$

For RBT3 we have:

$$u_c = z\frac{dw}{dx} = zA_1\frac{dh}{dx} \qquad (5)$$

Following Equations (4) and (5) analogously, shear deformation in-plane displacement for RBT1 and RBT2 is given as:

$$u_s = F(z).A_2\frac{dh}{dx} \quad (6)$$

A_2 was used in Equation (6) because shear deformation in-plane displacement is not equal to classical in-plane displacement $(u \neq u_c)$. For RBT2 F(z) = z.

By the analogy using Equation (4) the total in-plane displacement, u for RBT3 is:

$$u = zA_3\frac{dh}{dx} \quad (u_s \neq u_c) \quad (7)$$

For RBT1 the normal strain and vertical shear strain are:

$$\varepsilon_x = \frac{du}{dx} = \left(\frac{du_c}{dx} + \frac{du_s}{dx}\right) \quad (8)$$

$$\gamma_{xz} = \left(\frac{du_c}{dz} + \frac{du_s}{dz}\right) + \frac{dw}{dx} \quad (9)$$

Substituting Equations (1), (4) and (6) into Equations (8) and (9) gives:

$$\varepsilon_x = (-zA_1 + F(z)A_2)\frac{d^2h}{dx^2} \quad (10)$$

$$\gamma_{xz} = \left(-A_1 + \frac{dF(z)}{dz}A_2 + A_1\right)\frac{dh}{dx}. \text{That is,}$$

$$\gamma_{xz} = A_2\frac{dF(z)}{dz}\frac{dh}{dx} \quad (11)$$

For RBT2, the normal strain and vertical shear strain are:

$$\varepsilon_x = \frac{du}{dx} = \left(\frac{du_c}{dx} + \frac{du_s}{dx}\right) \quad (12)$$

$$\gamma_{xz} = \left(\frac{du_c}{dz} + \frac{du_s}{dz}\right) + \frac{dw}{dx} \quad (13)$$

Substituting Equations (1), (5) and (6) into Equations (12) and (13) gives:

$$\varepsilon_x = (A_1 + A_2).z\frac{d^2h}{dx^2} \quad (14)$$

$$\gamma_{xz} = (A_1 + A_2 + A_1)\frac{dh}{dx}. \text{That is,}$$

$$\gamma_{xz} = (2A_1 + A_2)\frac{dh}{dx} \quad (15)$$

For RBT3, the normal strain and vertical shear strain are:

$$\varepsilon_x = \frac{du}{dx} \quad (16)$$

$$\gamma_{xz} = \frac{du}{dz} + \frac{dw}{dx} \quad (17)$$

Substituting Equations (1), and (7) into Equations (16) and (17) gives:

$$\varepsilon_x = A_3.z\frac{d^2h}{dx^2} \quad (18)$$

$$\gamma_{xz} = (A_1 + A_3)\frac{dh}{dx}. \text{That is,}$$

$$\gamma_{xz} = (A_1 + A_3)\frac{dh}{dx} \quad (19)$$

Constitutive relations

The one dimensional constitutive law equations are:

$$\sigma_x = E\varepsilon_x \quad (20)$$

$$\tau_{xz} = \frac{E}{2(1+\mu)}\gamma_{xz} \quad (21)$$

Substituting Equations (10) and (11) into Equation (20) and (21) give for RBT1:

$$\sigma_x = E(-zA_1 + F(z)A_2)\frac{d^2h}{dx^2} \quad (22)$$

$$\tau_{xz} = \frac{EA_2}{2(1+\mu)}\frac{dF(z)}{dz}\frac{dh}{dx} \quad (23)$$

Substituting Equations (14) and (15) into Equation (20) and (21) give for RBT2:

$$\sigma_x = E(A_1 + A_2).z\frac{d^2h}{dx^2} \quad (24)$$

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$$\tau_{xz} = \frac{E}{2(1+\mu)}(2A_1 + A_2)\frac{dh}{dx} \qquad (25)$$

Substituting Equations (18) and (19) into Equation (20) and (21) give for RBT3:

$$\sigma_x = EA_3.z\frac{d^2h}{dx^2} \qquad (26)$$

$$\tau_{xz} = \frac{E}{2(1+\mu)}(A_1 + A_2)\frac{dh}{dx} \qquad (27)$$

Application of direct variational calculus

Substituting the various equations for ε_x, γ_{xz}, σ_x and τ_{xz} into Equation (1) gives total potential energy equations for RBT1, RBT2 and RBT3 respectively:

$$\Pi = \frac{D_1}{2}\int [\left({A_1}^2 - 2A_1A_2g_1 + {A_2}^2g_2\right)\left(\frac{d^2h}{dx^2}\right)^2 + \frac{{A_2}^2}{2(1+\mu)}\frac{\alpha^2 g_3}{L^2}\left(\frac{dh}{dx}\right)^2]dx - \int_x FF\,dx \quad (28)$$

$$\Pi = \frac{D_1}{2}\int [\left({A_1}^2 - 2A_1A_2 + {A_2}^2\right)\left(\frac{d^2h}{dx^2}\right)^2 + \frac{\left(24{A_1}^2 + 24A_1A_2 + 6{A_2}^2\right)}{(1+\mu)}.\frac{\alpha^2}{L^2}\left(\frac{dh}{dx}\right)^2]dx - \int_x FF\,dx \qquad (29)$$

$$\Pi = \frac{D_1}{2}\int [{A_3}^2\left(\frac{d^2h}{dx^2}\right)^2 + \frac{\left(6{A_1}^2 + 12A_1A_2 + 6{A_2}^2\right)}{(1+\mu)}.\frac{\alpha^2}{L^2}\left(\frac{dh}{dx}\right)^2]dx - \int_x FF\,dx \qquad (30)$$

$$\text{Where span} - \text{depth ratio,} \quad \alpha = \frac{L}{t}$$

$$D_1 = Eb\int_{-\frac{t}{2}}^{\frac{t}{2}} z^2\,dz = \frac{Ebt^3}{12}$$

$$g_1 = \frac{Eb}{D_1}\int_{-\frac{t}{2}}^{\frac{t}{2}} z.F(z)\,dz$$

$$g_2 = \frac{Eb}{D_1}\int_{-\frac{t}{2}}^{\frac{t}{2}} F(z)^2 dz$$

$$\frac{\alpha^2 g_3}{L^2} = \frac{Eb}{D_1}\int_{-\frac{t}{2}}^{\frac{t}{2}}\left[\frac{dF(z)}{dz}\right]^2 dz$$

Now let us define x coordinate in terms of non dimensional coordinate, R.

$$x = LR \qquad (31)$$

Equation (31) shall be substituted into Equations (28) and (29) and the resulting equations minimized with respect to A_1 and A_2 for RBT1 and RBT2 respectively. Also Equation (31) shall be substituted into Equation (30) and the resulting equation minimized with respect to A_1 and A_3 for RBT3. This minimizations resulted into two simultaneous equations for RBT1, RBT2 and RBT3 respectively:

$$\begin{bmatrix} k_1 & -g_1k_1 \\ -g_1k_1 & \left(g_2k_1 + \frac{g_3\alpha^2k_2}{2(1+\mu)}\right)\end{bmatrix}\begin{bmatrix}A_1\\A_2\end{bmatrix} = \frac{L^4}{D_1}\begin{bmatrix}C_1\\0\end{bmatrix} \qquad (32)$$

$$\begin{bmatrix} \left(k_1 + \frac{24\alpha^2k_2}{1+\mu}\right) & \left(k_1 + \frac{12\alpha^2k_2}{1+\mu}\right) \\ \left(k_1 + \frac{12\alpha^2k_2}{1+\mu}\right) & \left(k_1 + \frac{6\alpha^2k_2}{1+\mu}\right)\end{bmatrix}\begin{bmatrix}A_1\\A_2\end{bmatrix} = \frac{L^4}{D_1}\begin{bmatrix}C_1\\0\end{bmatrix} \qquad (33)$$

$$\begin{bmatrix} \left(k_1 + \frac{6\alpha^2 k_2}{1+\mu}\right) & \frac{6\alpha^2 k_2}{1+\mu} \\ \frac{6\alpha^2 k_2}{1+\mu} & \frac{6\alpha^2 k_2}{1+\mu} \end{bmatrix} \begin{bmatrix} A_3 \\ A_1 \end{bmatrix} = \frac{L^4}{D_1} \begin{bmatrix} 0 \\ C_1 \end{bmatrix} \quad (34)$$

Where

$$k_1 = \int_0^1 \left(\frac{d^2h}{dR^2}\right)^2 dR\,;\; k_2 = \int_0^1 \left(\frac{dh}{dR}\right)^2 dR$$

$$k_3 = \int_0^1 h\, dR; \qquad k_4 = \int_0^1 h^2 dR$$

For pure bending, buckling and free vibration, C1 is respectively:

$$C_1 = qk_3, \qquad C_1 = Nx\, k_2 \;\; and \;\; C_1 = \rho t \lambda^2 k_4$$

These three simultaneous equations shall resolved in simpler form for pure bending analysis, buckling analysis and free vibration analysis.

RBT1

Resolving Equation (32) gives (for pure bending analysis):

$$A_1 = \frac{k_3\left(g_2 k_1 + \frac{g_3\alpha^2}{1+\mu}k_2\right)}{{k_1}^2(g_2 - {g_1}^2) + \frac{g_3\alpha^2}{1+\mu}k_1k_2} \cdot \frac{qL^4}{D_1}$$

$$A_2 = \frac{g_1 k_3}{k_1(g_2 - {g_1}^2) + \frac{g_3\alpha^2}{1+\mu}k_2} \cdot \frac{qL^4}{D_1}$$

In the same way critical buckling load and fundamental natural frequencies can be obtained be resolving equation (32):

$$Nx = \frac{{k_1}^2(g_2 - {g_1}^2) + \frac{g_3\alpha^2}{1+\mu}k_1k_2}{k_2\left(g_2 k_1 + \frac{g_3\alpha^2}{2(1+\mu)}k_2\right)} \cdot \frac{D_1}{L^2}$$

$$\lambda^2 = \frac{{k_1}^2(g_2 - {g_1}^2) + \frac{g_3\alpha^2}{1+\mu}k_1k_2}{k_4\left(g_2 k_1 + \frac{g_3\alpha^2}{2(1+\mu)}k_2\right)} \cdot \frac{D_1}{\rho t L^4}$$

RBT2

Resolving Equation (33) gives (for pure bending analysis):

$$A_1 = \frac{k_3\left(k_1 + \frac{6\alpha^2}{1+\mu}k_2\right)}{\frac{6\alpha^2}{1+\mu}k_1k_2} \cdot \frac{qL^4}{D_1}$$

$$A_2 = -\frac{k_3\left(k_1 + \frac{12\alpha^2}{1+\mu}k_2\right)}{\frac{6\alpha^2}{1+\mu}k_1k_2} \cdot \frac{qL^4}{D_1}$$

In the same way critical buckling load and fundamental natural frequencies can be obtained be resolving equation (33):

$$Nx = \frac{\frac{6\alpha^2}{1+\mu}k_1k_2}{k_2\left(k_1 + \frac{6\alpha^2}{1+\mu}k_2\right)} \cdot \frac{D_1}{L^2}$$

$$\lambda^2 = \frac{\frac{6\alpha^2}{1+\mu}k_1k_2}{k_4\left(k_1 + \frac{6\alpha^2}{1+\mu}k_2\right)} \cdot \frac{D_1}{\rho t L^4}$$

RBT3

Resolving Equation (34) gives (for pure bending analysis):

$$A_1 = \frac{k_3\left(k_1 + \frac{6\alpha^2}{1+\mu}k_2\right)}{\frac{6\alpha^2}{1+\mu}k_1k_2} \cdot \frac{qL^4}{D_1}$$

$$A_2 = -\frac{k_3}{k_1} \cdot \frac{qL^4}{D_1}$$

In the same way critical buckling load and fundamental natural frequencies can be obtained be resolving equation (33):

$$Nx = \frac{\frac{6\alpha^2}{1+\mu}k_1k_2}{k_2\left(k_1 + \frac{6\alpha^2}{1+\mu}k_2\right)} \cdot \frac{D_1}{L^2}$$

$$\lambda^2 = \frac{\frac{6\alpha^2}{1+\mu}k_1k_2}{k_4\left(k_1 + \frac{6\alpha^2}{1+\mu}k_2\right)} \cdot \frac{D_1}{\rho t L^4}$$

Numerical example

Determine the center deflection - w, axial displacement at top of beam -u, critical buckling load -Nx and fundamental natural frequency λ^2 of a simply supported thick isotropic beam of rectangular cross section using Poisson's ratio of 0.3. The two deflection equations to be used are:

W = $A_1(R - 2R^3 + R^4)$ and w = A_1 Sin πR. For RBT1 (HSDT), the F(z) to be used is Krishna Murty Model (Krishna, 1984):

$$F(z) = \left[z - \frac{4z^3}{3t^2}\right]$$

For Krishna Murty Model, g1 = 0.8,

g2 = 68/105 and g3 = 6.4.

For Kaczkawski Model, g1 = 1, g = 1.0119 and g3 = 10.

Results and discussions

The results for center deflection, axial displacement, critical buckling load and fundamental natural frequencies are presented on tables 1, 2, 3 and 4.

All the value obtained from RBT2 tallied exactly with those from RBT3. Thus they were placed in the tables as RBT2/RBT3. So comparison was made between the values obtained from RBT1 and those from RBT2/RBT3 a critical look at the tables reveals closeness between the values from RBT1 and RBT2/RBT3. This closeness may be attributed

to the use of correct assumption that the engineering vertical shear strain is not zero by using $\gamma_{xz} = \left(\frac{dw}{dx} + \frac{du_s}{dz}\right) + \frac{dw}{dx}$ instead of using $\gamma_{xz} = \left(-\frac{dw}{dx} + \frac{du_s}{dz}\right) + \frac{dw}{dx}$. However, the limitation to the present theories RBT2 and RBT3 remains the assumption of uniform vertical shear stress across the cross section of the beam. The RB2 and RBT3 can be used in high confidence for analysis of beams of various boundary conditions by simply applying the formulas derived herein. It is very obvious that the present methods are straight forward and present no difficulty and are not rigorous in application. This paper is making recommendation for the extension of this present study to those of plate problems.

Table 1: Center deflection, $w\ (qL^4/D_1)$

	POLYNOMIAL			TRIGONOMETRY		
L/t	RBT1	RBT2/RBT3	%Diff	RBT1	RBT2/RBT3	%Diff
100	0.013024	0.013024	0.0000	0.013074	0.013074	0.0000
30	0.013058	0.013052	0.0460	0.013108	0.013102	0.0458
20	0.013105	0.013091	0.1069	0.013155	0.013141	0.1065
15	0.01317	0.013145	0.1902	0.01322	0.013195	0.1895
10	0.013355	0.0133	0.4135	0.013406	0.013351	0.4120
4	0.015108	0.014763	2.3369	0.015163	0.014818	2.3282

Table 2: Axial displacement, $u\ (qL^3t/D_1)$

	POLYNOMIAL			TRIGONOMETRY		
L/t	RBT1	RBT2/RBT3	Diff	RBT1	RBT2/RBT3	%Diff
100	-0.020833	-0.020833	0.0000	-0.020532	-0.020532	0.0000
30	-0.020831	-0.020833	-0.0096	-0.02053	-0.020532	-0.0097
20	-0.020828	-0.020833	-0.0240	-0.020526	-0.020532	-0.0292
15	-0.020823	-0.020833	-0.0480	-0.020522	-0.020532	-0.0487
10	-0.020811	-0.020833	-0.1056	-0.02051	-0.020532	-0.1071
4	-0.020694	-0.020833	-0.6672	-0.020395	-0.020532	-0.6673

Table 3: Critical buckling load, $N_x\ (\frac{D_1}{L^3})$

	POLYNOMIAL			TRIGONOMETRY		
L/t	RBT1	RBT2/RBT3	Diff	RBT1	RBT2/RBT3	%Diff
100	9.879814	9.880237	-0.0043	9.867072	9.867494	-0.0043
30	9.854221	9.858898	-0.0474	9.841545	9.84621	-0.0474
20	9.819283	9.829735	-0.1063	9.806697	9.817122	-0.1062
15	9.77079	9.789196	-0.1880	9.758327	9.776686	-0.1878
10	9.634869	9.67519	-0.4167	9.622751	9.66297	-0.4162
4	8.517201	8.715953	-2.2803	8.507726	8.706035	-2.2778

Table 4: Fundamental natural frequency, $\lambda \left(\sqrt{\frac{D_1}{\rho t L^4}}\right)$

	POLYNOMIAL			TRIGONOMETRY		
L/t	RBT1	RBT2/RBT3	Diff	RBT1	RBT2/RBT3	%Diff
100	9.874677	9.875602	-0.0094	9.867626	9.868549	-0.0094
30	9.854684	9.864931	-0.1039	9.847673	9.8579	-0.1037
20	9.827366	9.85033	-0.2331	9.82041	9.843328	-0.2328
15	9.789398	9.829997	-0.4130	9.782518	9.823035	-0.4125
10	9.682664	9.772589	-0.9202	9.675993	9.765741	-0.9190
4	8.78428	9.2755	-5.2959	8.779252	9.269581	-5.2897

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